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TABLE OF CONTENTS

Foundations 1

Algebra 3

Abstract algebra 5

Theory of groups 7

Number theory 11

Analysis 16

Theory of sets, theory of functions of real variables 17

Theory of functions of complex variables . 20

Theory of series 25

Fourier series and generalizations, integral transforms 26

Polynomials, polynomial approximations . 29

Special functions 30

Differential equations 31

Integral equations 37

Functional analysis, ergodic theory . . . 38

Theory of probability 40

Mathematical statistics 41

Topology 43

Geometry 49

Convex domains, extremal problems . . . 51

Differential geometry 52

Numerical and graphical methods . . . 56

Relativity 59

Mechanics 60

Hydrodynamics, aerodynamics 61

Elasticity, plasticity 67

AUTHOR INDEX

Abellanas, P. 5
Agmon, S. 17
Agranović, Z. 28
Aigner, A. 3
Albert, A. A. 6
Al'per, S. Y. 24
Altwegg, M. 18
Alumyaè, N. A. 70
Amerio, L. 36
Andrews, F. C. See Birnbaum, Z. W.
Ankeny, N. C. 11
Arelkin, G. Y. 19, 43
Arghiriade, E. 52
Arżanyh, I. S. 35
Asano, K. 5
Atkinson, F. V.-Cherwell. 15
Aubert, M. See Guillet, A.
Bailey, W. N. 64
Balachandran, V. K. 5
Bancroft, T. A. 30
Banerjee, D. P. 41
Barajas, A. 60
Barbuti, U. 32
Bari, N. K. 26
Batchelor, G. K. 63
Behari, R.-Mishra, R. S. 53
Bellman, R. 31
Bernštein, S. N. 22, 23
Besicovitch, A. S. 51
Binnie, A. M. 61
Birindelli, C. 25
Birman, S. E. 68
Birnbaum, Z. W.-Andrews, F. C. 41
Blakers, A. L.-Massey, W. S. 47
Blambert, M. 20
Blaschke, W. 50
Boas, R. P. 27
Bogolyubov, N. N.-Hacet, B. I. 40
von Bohl, J. G. 30
Bol, G. 52, 53
Boltyanskiĭ, V. 45
Booth, A. D.-Britten, K. H. V. 58
Borodin, V. A.-Dityakin, Y. F. 66
Bose, S. K. 28
van Bouchout, V. 52
Brändli, E. R. 47
Britten, K. H. V. See Booth, A. D.
Brodskiĭ, M. L. 18
Brouwer, L. E. J. 2
Bruk, I. S. 58
Bückner, H. 58
Busemann, H. 56
Čakalov, L. N. 27
Calabi, L.-Ehresmann, C. 9
Calder, K. L. 66
Cârstoiu, I. 61
Cartan, H. 46
Cartwright, M. L.-Copson, E. T.-Greig, J. 32
Castoldi, L. 60
Cesari, L. 20
Chang, Shih-Hsun. 38
Charles, F. 17
Charles, H. 35
Chaundy, T. W. 31
Cherry, T. M. 28, 34
Cherwell. See Atkinson, F. V.
Choquet, G. 17
Chowla, S. D.-Todd, J. 14
Clark, G. L. 59
Claudian, V. 11
Cohen, L. W.-Goffman, C. 44
Colombo, G. 30, 33
Combes, J. 24
Comrie, L. J. 56
Conforto, F. 24
Conti, R. 19
Copson, E. T. See Cartwright, M. L.
Craig, H. V. 60
Danford, M. B. See Greenwood, R. E.
Davenport, H. 15
Deaux, R. 51
DeBaggis, H. F. 50
Dedò, M. 4
Denjoy, A. 17
Dienes, P. 1
Dienes, Z. P. 1
Dinghas, A. 22
Dingler, H. 2
Di Noi, S. 50
Dityakin, Y. F. See Borodin, V. A.
Doob, J. L. 43
Dörr, J. 60
Dronkers, J. J. 36
Drozdovič, V. N. See Vorob'ev, Y. V.
Duncan, W. J. 60
Dvuhèerstov, G. I. 66
Džanelidze, G. Y. 68
Ehresmann, C. See Calabi, L.
Eilenberg, S. 8
Einstein, A.-Infeld, L. 59
Ellis, D. 9
Erdös, P. 40
Erdös, P.-Gál, I. S. 14
Erdös, P.-Koksma, J. F. 14
Ertel, H.-Köhler, H. 62
Fast, H. See Hartman, S.
Federhofer, K. 71
Fedorov, V. S. 18
Fenyö, S. 16
Ferrier, A. 11
Fet, A. 47
Feys, R. 1
Finney, D. J. 42
Fitch, F. B. 2
Foulkes, H. O. 4
Fox, R. H. 45
Fraïssé, R. 17
Frame, J. S. 4
Fréchet, M. 40, 42
Frocht, M. M. 58
Fuchs, L. 9
Gál, I. S. See Erdös, P.
Galin, L. A. 70
Gambier, B. 51
Garabedian, P. R. 21
Garrick, I. E.-Rubinow, S. I. 65
Gavrilov, M. A. 3
Geldel'man, R. M. 53
Germay, R.-H.-J. 21, 38
Ghizzetti, A. 37
Ghurye, S. G. 31
Gilbarg, D. 61
Goffman, C. See Cohen, L. W.
Gobeen, H. E. 37
Gonzalez Baz, E. 26

(*Continued on cover 4*)

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FOUNDATIONS

Feys, Robert. A simple notation for relations. Methodos 1, 79–93 (1949).

This work gives a simpler and more uniform set of definitions of the notions for the theory of relations as developed in Principia Mathematica. The notions in question include the fundamental ideas of what is now called the "calculus of relations" (namely: union, intersection, complement, converse, and relative product) and in addition certain ideas connecting relations with classes (domain, converse domain, field, and the operation of restricting the domain or converse domain of a given relation). The simplification rests in part on the use of the notations of combinatory logic.

It appears to the reviewer that it might have been preferable, in connection with definition 8.11, to put some restriction on the variable β: perhaps to say that β is the first variable, in the sequence of all variables, which is different from α and does not occur in Δ. In 4.17, the letter "r" should be inserted before the "x." In line 5 of page 87, the symbols "$\sim$" and "$\hat{x}$" should be transposed. In 7.41, the symbol "$\hat{x}$" should be replaced by "$\hat{y}$"; similarly on page 88, line 7 from the bottom. On line 18 of page 91, the letter "h" should be replaced by "k." *J. C. C. McKinsey.*

Wendelin, H. Ein Kriterium für die Erweiterbarkeit einer Implikation zu einer Äquivalenz. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. 85, 66–69, 216 (1948).

This is a proof of a theorem which can also be expressed in the following way: if f and g are functions such that, for all x and y, $y=f(x)$ implies $y=g(x)$, then f and g are identical. *J. C. C. McKinsey* (Stillwater, Okla.).

Dienes, Paul. On ternary logic. J. Symbolic Logic 14, 85–94 (1949).

Let the ordinary matrix definition of binary logic be extended by adjunction of the value $\frac{1}{2}$. The author investigates how the matrix must be filled up to satisfy various conditions, e.g., associative disjunctions and conjunctions, mutually distributive pairs of associative disjunctions and conjunctions. He further investigates the "de Morgan transform" $\bar{f}(\bar{p}, \bar{q})$ of a function $f(p, q)$, where the negation function denoted by the bar may be chosen in different ways. For several formulae it is indicated for which matrices they are identically 1. [Reviewer's remark. The system considered in note 12 is not Jaśkowski's characteristic matrix of intuitionistic logic; in the latter ($p \rightarrow q$ or $q \rightarrow p$) is not identically 1.] *A. Heyting* (Amsterdam).

Dienes, Z. P. On an implication function in many-valued systems of logic. J. Symbolic Logic 14, 95–97 (1949).

Let A be an arbitrary set of numbers in the interval $0 \leqq x < \frac{1}{2}$, B the set of numbers $1-x$, where x is in A. We define $Nx = 1-x$, $\pi xy = \max(x, y)$, $\sigma xy = N\pi NxNy = \min(x, y)$, $C'xy = \pi Nxy = \max(1-x, y)$. Then $A \cup B$ is a characteristic matrix of the calculus of propositions, with B as the set of designated elements. *A. Heyting* (Amsterdam).

Novikov, P. S. On classes of regularity. Doklady Akad. Nauk SSSR (N.S.) 64, 293–295 (1949). (Russian)

In an earlier paper [Rec. Math. [Mat. Sbornik] N.S. 12(54), 231–261 (1943); these Rev. 5, 197] the author defined "classes of regularity" in certain logics containing infinite sums and products. He now gives definitions suitable for finite systems, and applies them to independence proofs. In this paper the logical system is the predicate calculus, without quantifiers for predicates, but with symbols for object-functions as well as object-individuals. An "elementary formula" is one containing no logical operations [i.e., presumably, a proposition-symbol or a predicate-symbol with arbitrary terms as arguments]; and an "elementary sum" is a logical sum of elementary formulae or their negations. Any class of elementary sums is a "class of primitive formulae," provided that it is closed under the operation of replacing an object-variable by any term, and that it contains all elementary sums in which one summand is the negation of another. The "class of regularity" determined by a class II of primitive formulae is the set of all formulae, the reduced form [see below] of which can be transformed into a member of II by repeated application of the following three operations. Let the given formula have the form $\prod_i \sum_j \mathfrak{A}_{ij}$. Operation 1: if an $\mathfrak{A}_{ij}$ is $(x)\mathfrak{B}(x)$, omit the quantifier and replace x by a new free variable; operation 2: if $\mathfrak{A}_{ij}$ is $(Ex)\mathfrak{B}(x)$, omit (Ex) and replace x by an arbitrary term, subject to avoiding accidental clashes; operation 3: apply the distributive law (for $(\mathfrak{A} \,\&\, \mathfrak{B}) \vee \mathfrak{C}$) of the algebra of logic. By the "reduced form" of a formula the author appears to mean the form containing only $\&$, $\vee$, $^-$ and quantifiers, in which all negations are applied directly to propositional and predicate-variables (though he says "containing no logical operations except $\&$ and $^-$"). The following theorem is announced [without proof]. Every formula deducible in the predicate calculus from the formulae of a class of regularity belongs to that class.

If we take the system to be the predicate calculus with the axioms of arithmetic adjoined (including recursive definitions) and as a class of primitive formulae the class of all provable elementary logical sums, the class of regularity contains all the axioms except the induction axiom, proving its independence; and this remains true if further (consistent) axioms are adjoined not containing predicate variables. These, as the author notes, are not new results, but new applications are promised in a further paper. The notion of class of regularity can be extended to other logical systems, e.g., the simple theory of types.

M. H. A. Newman (Manchester).

Hall, Marshall, Jr. The word problem for semigroups with two generators. J. Symbolic Logic 14, 115–118 (1949).

"Semi-group" has here the sense "associative multiplicative system" [without the requirement of a cancellation law added by some writers]. A semi-group with only two generators having an unsolvable word problem is con-

structed, by shewing how to reduce the word-problem for any "Thue system" [E. L. Post, same J. 12, 1–11 (1947); these Rev. 8, 558] to the word-problem for a Thue system on two letters. The result then follows from Post's theorem for Thue systems [op. cit.]. *M. H. A. Newman.*

Fitch, Frederic B. On natural numbers, integers, and rationals. J. Symbolic Logic **14**, 81–84 (1949).

A theory of natural numbers is outlined using the simple theory of types as underlying logic. Given a class M of relations $A, B, \cdots$, the relation of the Nth power of A to A (i.e., the class of ordered pairs (A^N, A)) is regarded as the natural number N. The usual operations on these are defined. An extension to the rationals is also described and there is a brief indication of an extension to the reals. The author hopes to make use of this theory in connection with his system of "basic logic" and suggests that it might also have some useful connection with the theory of groups and fields. *I. L. Novak* (Wellesley, Mass.).

Hinčin, A. Ya. The simplest linear continuum. Uspehi Matem. Nauk (N.S.) **4**, no. 2(30), 180–197 (1949). (Russian)

The author sketches an axiomatic foundation of analysis. The axioms are as follows. (I) The continuum K is a linearly ordered set. (II) Between any two different elements of K there exists another element of K. (III) (axiom of continuous induction). If a property S of elements of K satisfies the conditions: (1) there exists at least one element α such that every element less than α possesses S, (2) if every element less than β possesses S, there exists at least one element $\gamma > \beta$ such that every element less than γ possesses S, then every element of K possesses S. On the basis of (I) and (II), (III) is equivalent to the Dedekind axiom. For the first part of the proof, let Dedekind be valid and S denote a property satisfying (1) and (2). An element λ is assigned to the class A if every element less than λ possesses S, otherwise to B. Suppose B is not empty; the cut (A, B) defines an element α. If $\alpha \varepsilon B$, there would exist an element $\beta < \alpha$ not possessing S, and consequently an element γ not possessing S, such that $\beta < \gamma < \alpha$, which contradicts $\gamma \varepsilon A$. But if $\alpha \varepsilon B$, condition (2) leads to a contradiction. For the second part of the proof, let (III) be valid and suppose the Dedekind cut (A, B) defines no element of K. The property S of belonging to A satisfies condition (1) of (III). If $\beta \varepsilon A$, there exists a $\gamma > \beta$, so that $\gamma \varepsilon A$, and every element between β and γ will also belong to A, so S also fulfills condition (2). Then B is empty, contrary to assumption. The axioms (I), (II), (III) allow the development of the topological part of analysis; this possibility is one of the advantages of the axiomatic foundation. In order to characterize the ordinary (simplest) continuum, the author introduces axiom (IV): there exists a continuous function $\varphi(x, y)$ so that from $x < y$ follows $x < \varphi(x, y) < y$ [$\varphi(x, y)$ is supposed to be continuous and to satisfy the condition also for $x, y = \pm\infty$]. Now a denumerable set which is dense in K is constructed as follows: $a_0 = -\infty$, $a_1 = +\infty$, if $\alpha = a_{2^{-k}}$ and $\beta = (a+1)2^{-k}$, then $\varphi(a_\alpha, a_\beta) = a_{\frac{1}{2}(\alpha+\beta)}$. Here a_α and a_β are of rank k, $a_{\frac{1}{2}(\alpha+\beta)}$ of rank $k+1$. Let D be the set of elements a_λ determined in this way and D^* its closure. Suppose D^* does not coincide with K and let (a, b) be an interval contiguous to D^*. Then (a, b) is contained in a minimal interval (r_n, s_n), whose endpoints are of rank n. It is easily seen that $\lim r_n = a$, $\lim s_n = b$, so $\lim \varphi(r_n, s_n) = \varphi(a, b) = c$; but $\varphi(r_n, s_n)$ and consequently c cannot fall within (a, b), a result which implies a contradiction. *A. Heyting* (Amsterdam).

Brouwer, L. E. J. Contradictoriness of elementary geometry. Nederl. Akad. Wetensch., Proc. **52**, 315–316 = Indagationes Math. **11**, 89–90 (1949). (Dutch)

By means of the ideas and methods described in two previous papers [same Proc. **51**, 963–964 (1948); **52**, 122–124 (1949) = Indagationes Math. **10**, 322–323 (1948); **11**, 37–39 (1949); these Rev. **10**, 421, 499] a contradiction is obtained from the proposition "Any two straight lines in the real projective plane, which cannot coincide, possess a point of intersection." *A. Heyting* (Amsterdam).

Dingler, Hugo. Betrachtungen zur Axiomatik. Methodos **1**, 1–33 (1949).

The history of methodical reasoning is traced in Greek thought from mythology to Euclid. The basic properties of an axiom system are described, and errors of application are listed. Limitations of the axiomatic method are indicated, and the matter of interpretation is considered. The treatment is elementary and expository.

C. C. Torrance (Annapolis, Md.).

***Whittaker, Edmund. From Euclid to Eddington. A Study of Conceptions of the External World.** Cambridge University Press, 1949. ix+212 pp. $4.00.

This book sketches the development of the fundamental concepts of physical science. The notion of space is traced from Plato, via Gassendi and Kant, to some of its present forms. Euclid's ideas of point, line and proof are outlined and criticized, with an indication given of modern treatments. Tests are described for determining the Euclidean or non-Euclidean nature of physical space. Two scales of time are discussed, the law of composition of velocities is derived by an ingeniously simple method, and the principles of special relativity are outlined. The classical concepts of mass, energy, etc., are traced from William of Ockham. Various aether theories are reviewed, and the mass-energy relation is developed.

The second half of this book is an introductory exposition of modern ideas. The physical principles of general relativity are lucidly stated, and some of their consequences are indicated; unified theories are appraised. Physical relations involving Planck's constant h are described, wave and matrix mechanics are outlined, and the idea of spin is carefully presented in an elementary way. Eddington's theories and results are summarized.

This book is an admirable example of simple technical exposition, for the author presents experimental evidence, develops concepts, and draws conclusions in a very intelligible manner. But unfortunately, the author's philosophical views are not clear. On p. 16 he says that "it would be unprofitable to carry the analysis [of the spatial concept] further. . . . There is then no difficulty in building up the system of physical knowledge." And yet on p. 112 he points out that "the fundamental physical problems connected with it [gravity] are almost as perplexing to-day as they have ever been," and on p. 199 he quotes Eddington's view that "exclusion is simply the quantum-mechanical aspect of . . . gravitation." Perhaps the author, in attempting to bypass philosophical difficulties, did not consider that the object of abstraction is to ignore and explain reality. In any event, the empirical facts of the subject are presented most lucidly and appealingly. *C. C. Torrance.*

ALGEBRA

Aigner, Alexander. Gerade und ungerade Permutationen in geordneter Reihenfolge. Elemente der Math. **4**, 52–54 (1949).

If permutations of 1 to n are put in numerical order, the one having number N is odd or even according as k is odd or even, $k=k_1+k_2+\cdots+k_{n-1}$ and

$$N-1=k_1(n-1)!+k_2(n-2)!+\cdots+k_{n-1}.$$

J. Riordan (New York, N. Y.).

Kreis, H. Über die Potenzsummen der natürlichen Zahlen. Elemente der Math. **4**, 54–56 (1949).

Proof of known results [Nielsen, Traité Élémentaire des Nombres de Bernoulli, Gauthier-Villars, Paris, 1923, p. 61] on the sums of powers $S_p(n)=1^p+2^p+\cdots+n^p$; namely (i) $S_{2q}(n)/S_2(n)$ is a polynomial of degree $q-1$ in $(2n+1)^2$ and (ii) $S_{2q+3}(n)/S_3(n)$ is a polynomial of degree q in $(2n+1)^2$.

J. Riordan (New York, N. Y.).

Zahlen, Jean-Pierre. Sur une généralisation d'un problème d'Euler. Euclides, Madrid **9**, 45–47 (1949).

A complete solution of the following generalization of a game problem given by Euler: there are n players and n turns; player r loses on the rth turn the amount necessary to multiply by d the holdings of all other players at the start of the turn; if at the finish all players hold the same amount, what are their initial holdings? *J. Riordan.*

Gavrilov, M. A. Relay-contact decipherers using several impulse counters. Avtomatika i Telemehanika **10**, 157–183 (1949). (Russian)

Formulas are derived for the number of relay contacts and contact springs needed in numerous relay circuits designed to decipher messages encoded as sequences, z digits long, of digits to the base n. *E. N. Gilbert.*

Schützenberger, M. P. A non-existence theorem for an infinite family of symmetrical block designs. Ann. Eugenics **14**, 286–287 (1949).

The author shows that a symmetrical balanced incomplete block design is impossible if v, the number of varieties, is even and $r-\lambda$ is not a square, where r is the number of times each variety occurs and λ the number of times each pair of varieties occurs together in a block. This is shown by computing the determinant of the incidence matrix, $|A|=r(r-\lambda)(v-1)/2$. The author's proof can be considerably simplified by computing $|AA^T|$ instead of $|A|$ directly.

H. B. Mann (Berkeley, Calif.).

Harshbarger, Boyd. Triple rectangular lattices. Biometrics **5**, 1–13 (1949).

The design considered by the author can be constructed when an m-sided Latin square is available for which the numbers in the principal diagonal are all different. One then constructs from this Latin square a triple lattice in the usual manner and deletes in each block the variety v_{iii} occurring in it. Thus three sets of m blocks each are obtained, each block containing $(m-1)$ varieties. The author gives the analysis of these designs utilizing the interblock information and illustrates the procedure by an example. *H. B. Mann* (Berkeley, Calif.).

Vinograde, B. An application of Newton's power-sum formulas. Amer. Math. Monthly **56**, 377–379 (1949).

The author discusses the conditions upon two real polynomials $f(x)$ and $g(x)$ of equal degree such that $f(g(x))=g(f(x))$. By use of Newton's formulas expressing the sums of the powers of the roots in terms of the elementary symmetric functions of the roots, the author is able to show that $f(x)=g(x)$, unless $f(x)$ has the form $f(x)=c+\sum_{j=1}^{k}a_j(x-c)^{2j-1}$, when $g(x)=-f(x)+2c$. This result leads to the theorem that any family of real mutually commutative polynomial operators with at least one operator of each degree has exactly one operator of each degree and is determined by the prime degree operators. *M. Marden.*

Parodi, Maurice. Sur une méthode de formation du polynome ayant pour racines les sommes des racines de deux polynomes donnés. C. R. Acad. Sci. Paris **228**, 1911–1913 (1949).

The required polynomial is the characteristic function of the matrix $M\langle I_n\rangle+I_m\langle N\rangle$, where I_r is the unit matrix of order r, where M, N are matrices of orders m, n whose characteristic functions are the given polynomials, and where $A\langle B\rangle$ denotes the direct product of the matrices A and B.

D. E. Rutherford (St. Andrews).

Schmidt, Hermann. Ein Gegenstück zu den Unzerlegbarkeitssätzen von Eisenstein und Dumas für den Fall einer beliebigen Bewertung. Math. Nachr. **2**, 1–3 (1949).

Let $f(x)=c_0x^n+\cdots+c_n$ be a polynomial, with coefficients in a field with a valuation V, and having a linear Newton diagram. If f has a factor of degree k, then $k(Vc_n-Vc_0)/n$ is in the value group. An extension is given to the case of a broken diagram. These results are implicit in a paper by MacLane [Trans. Amer. Math. Soc. **43**, 226–239 (1938)], apparently not available to the author. *I. Kaplansky.*

Korevaar, J., and Scheelbeek, P. A. J. Inversion of a matrix. Math. Centrum Amsterdam. Rapport **ZTW 1949-001**, 5 pp. (1949). (Dutch)

The matrix in question is a $t+1$ by $t+1$ matrix with $\binom{n+i+j-1}{i+j}$ as the element in the jth row and ith column. A formula, too complicated to be reproduced here, is obtained for the cofactor of $\binom{n+i+j}{i+j}$. *A. W. Goodman.*

Taussky, Olga. On a theorem of Latimer and MacDuffee. Canadian J. Math. **1**, 300–302 (1949).

Let $f(x)$ be an irreducible polynomial of degree n with rational integral coefficients and let $f(X)=0$, where X is an $n\times n$ matrix with rational integral elements. Then $X\omega=\alpha\omega$, where α is a numerical root of $f(x)=0$ and the components of the column vector ω can be chosen to form the basis of an ideal in the ring generated by α. The correspondence thus established is one-to-one between ideals and classes of equivalent matrices, X being equivalent to SXS^{-1}, where S has rational integral elements and determinant ±1. Under the restriction to irreducible $f(x)$, a new proof is given of these results, which are due to Latimer and MacDuffee [Ann. of Math. (2) **34**, 313–316 (1933)].

W. Givens (Knoxville, Tenn.).

Hayashida, Tsuyoshi. On faithful representations of free groups. Kōdai Math. Sem. Rep., no. **2**, 27 (1949).

A short proof is given of the fact that no identity of the form $A^{\alpha_1}B^{\beta_1}\cdots L^{\lambda_1}\cdots L^{\lambda_j}=E$, with $\alpha_1,\cdots,\lambda_j$ nonzero integers, holds for all choices of nonsingular matrices $A, B, \cdots, L$ over any infinite field. The general theory is reduced to the case of two-rowed matrices, and it is shown

that the two-rowed modular group contains as a subgroup every free group with a countable number of generators. *R. M. Thrall* (Ann Arbor, Mich.).

Gyires, Béla. Funktionensysteme mit vertauschbaren Gramschen Matrizen. Hungarica Acta Math. 1, no. 4, 28–32 (1949).

Routine matrix methods are used in finding all real symmetric positive definite matrices G which commute with a given matrix F satisfying the same conditions. *W. Givens* (Knoxville, Tenn.).

Gyires, Béla. Darstellung symmetrischer regulärer Matrizen als Produkt von zueinander transponierten Matrizen. Hungarica Acta Math. 1, no. 4, 33–35 (1949).

The general solution of the matrix equation $XX'=F$, where F is a real symmetric positive definite matrix, is studied and familiar results are obtained. *W. Givens.*

Frame, J. S. Congruence relations between the traces of matrix powers. Canadian J. Math. 1, 303–304 (1949).

The author proves the following theorem. Let $S(m)$ = trace of A^m, where A is a matrix of finite order n and finite degree $S(0)$, and assume that A is such that $S(k)=S(1)$ for all k such that $(k, n)=1$. Then $S(p^\beta)\equiv S(p^{\beta-1})$ (mod p^β), where p^β is a prime power greater than 1. An application is given to the problem of constructing the table of characters of a finite group. *W. Givens* (Knoxville, Tenn.).

Suleĭmanova, H. R. Stochastic matrices with real characteristic numbers. Doklady Akad. Nauk SSSR (N.S.) 66, 343–345 (1949). (Russian)

A matrix (p_{ij}) is a stochastic matrix if its elements are nonnegative and if the row sums are 1; it is called positive if in addition no element vanishes. Several theorems are proved on the range of variation of the characteristic values of stochastic matrices. For example: the n real numbers $1, \lambda_1, \cdots, \lambda_{n-1}$ with $|\lambda_j|<1$ are the characteristic values of some positive stochastic matrix of order n if $\sum|\lambda_j|<1$, where the sum is over the j's with $\lambda_j<0$. If the λ_j's are all negative the condition is also necessary. *J. L. Doob.*

Wielandt, Helmut. Ein Einschliessungssatz für charakteristische Wurzeln normaler Matrizen. Arch. Math. 1, 348–352 (1949).

Let L be a normal matrix of order n over the complex field, y and $z=Ly$ column vectors, $m_{00}=(y, y)$, $m_{01}=(z, y)=\bar{m}_{10}$, $m_{11}=(z, z)$, where $(y, z)=\sum\eta_\nu\bar{\zeta}_\nu$ is the usual Hermitian inner product of the vectors y and z, and call

$$\langle f\rangle=\alpha_{00}m_{00}+\alpha_{01}m_{01}+\alpha_{10}m_{10}+\alpha_{11}m_{11}$$

the moment of the "bipolynomial"

$$f(\lambda)=\alpha_{00}+\alpha_{01}\lambda+\alpha_{10}\bar{\lambda}+\alpha_{11}\lambda\bar{\lambda}.$$

If $f(\lambda)$ is a bipolynomial such that the real part of $\langle f\rangle$ is $\geqq 0$, then L has an eigenvalue λ_ν such that the real part of $f(\lambda_\nu)$ is $\geqq 0$. The proof of this theorem is immediate if y is expressed as a linear combination of a complete orthonormal system of eigenvectors of L. The author shows that suitable specializations of $f(\lambda)$ yield the generalizations to normal matrices of four known results on real symmetric matrices. A geometric summary is given: a closed circular region which contains the opposite ends of a diameter of the circle with center $\rho=(m_{01}/m_{00})^{\frac{1}{2}}$ and radius $\delta=(m_{11}/m_{00}-\rho\bar{\rho})^{\frac{1}{2}}$ must contain an eigenvalue of L. In this form the result is identical with a theorem of A. G. Walker and J. D. Weston [J. London Math. Soc. 24, 28–31 (1949); these Rev. 10, 501]. *W. Givens* (Knoxville, Tenn.).

Raymond, François H. Remarque sur la stabilité en connexion avec les valeurs propres d'une matrice. C. R. Acad. Sci. Paris 228, 1564–1565 (1949).

The author states the theorem that, if a matrix A of order n is such that all the elements of the matrix $B=(A-I)^{-1}(A+I)$ are in modulus less than $1/n$, then all the characteristic values of A have negative real parts. The proof is based upon the transformation $w=n(z-1)^{-1}(z+1)$ carrying the half plane $\Re(z)<0$ into the circle $|w|<n$ and the theorem that, if a matrix of order n has only elements of modulus less than one, its characteristic values lie in the circle $|w|<n$. Reference is made to the note of Wegner [same vol., 1200 (1949); these Rev. 10, 586]. *M. Marden* (Milwaukee, Wis.).

Dedò, Modesto. Sulle trasformazioni quadratiche e su alcuni procedimenti classici in cui esse si presentano implicitamente. Period. Mat. (4) 27, 73–88 (1949).

Foulkes, H. O. Differential operators associated with S-functions. J. London Math. Soc. 24, 136–143 (1949).

Hammond and Macmahon have employed extensively certain differential operators in the theory of symmetric functions. Since then, however, there have been defined the new type of symmetric functions called S-functions, related to other symmetric functions by the equations

$$\{\lambda\}=\{\lambda_1, \cdots, \lambda_p\}=\frac{1}{n!}\sum h_\rho\chi_\rho^{(\lambda)}S_1^{\rho}S_2^{\rho}S_3^{\rho}\cdots$$

$$=|h_{\lambda_r-r+t}|=|a_{\mu_s-s+t}|=|\alpha_s^{\lambda_t+n-t}|\div|\alpha_s^{n-t}|.$$

The author defines corresponding differential operators

$$D_\lambda=\frac{1}{n!}\sum h_\rho\chi_\rho^{(\lambda)}\frac{\partial^{\sigma+\tau+\omega+\cdots}}{\partial S_1^{\sigma}\partial S_2^{\tau}\partial S_3^{\omega}\cdots}.$$

These operators are connected by equations similar to those for S-functions. Thus

$$\{21\}=h_2h_1-h_3=a_2a_1-a_3$$

and correspondingly

$$D_{21}=D_2D_1-D_3=D_1^2D_1-D_1^3.$$

Thence it is shown that

$$D_\lambda\{\mu\}=\sum g_{\lambda\nu\mu}\{\nu\}=|h_{\lambda_r-\lambda_t-r+t}|,$$

where $g_{\lambda\nu\mu}$ is the coefficient of $\{\mu\}$ in the product $\{\nu\}\{\lambda\}$. The author discusses the effect of operating on $\{\lambda\}^r$ and shows how the results can be used to facilitate the evaluation of the operation $\{\lambda\}\otimes\{\mu\}$ which has important applications in invariant theory. *D. E. Littlewood* (Bangor).

Foulkes, H. O. A note on S functions. Quart. J. Math., Oxford Ser. 20, 190–192 (1949).

The paper gives a rule for expressing S-functions of the variables $x_i, \epsilon x_i, \cdots, \epsilon^{r-1}x_i$ for $i=1, 2, \cdots, n$ and $\epsilon^r=1$, in terms of S-functions of x_i^r. In the language of the symmetric group rather than the full linear group, this is just the problem of expressing the characters of elements of S_{nr}, each of whose cycles is divisible by r, in terms of the characters of S_n. As such, the result is a special case of a general formula given by the reviewer [Amer. J. Math. 70, 277–294 (1948); these Rev. 10, 678] and based on the notion of the

star diagram $[\alpha]^*$ of a given diagram $[\alpha]$, which makes the expression for the characters in question explicit.

G. de B. Robinson (Toronto, Ont.).

Abstract Algebra

Hashimoto, Junji. On the product decomposition of partially ordered sets. Math. Japonicae 1, 120–123 (1948).

It is shown that the unique factorization theorem is false for partially ordered sets P in general (it is true if P has a 0 and a I). A substitute for the theorem is suggested. The essential idea is a new correlation between factorizations of P, and factorizations of polynomial forms with nonnegative coefficients. *G. Birkhoff* (Cambridge, Mass.).

Pickert, Günter. Zur Übertragung der Kettensätze. Math. Ann. 121, 100–102 (1949).

It is shown that if M is a modular lattice in which $I=a\cup b$, and if the intervals $a/0$ and $b/0$ satisfy the ascending or descending chain condition, then $I/0$ satisfies it also. This generalizes a result of E. Noether. *G. Birkhoff.*

Balachandran, V. K. Prime ideals and the theory of last-residue-classes. J. Indian Math. Soc. (N.S.) 13, 31–40 (1949).

The last residue class of an α-ideal A in a distributive lattice L is the set of $x\varepsilon L$ such that $x\cap a=0$ for some $a\varepsilon A$ (in Birkhoff's terminology, μ-ideal = ideal, α-ideal = dual ideal). The last residue class of an α-ideal A is the intersection of all minimal-prime μ-ideals disjoint with A. An α-ideal A is the last residue class of some μ-ideal B if and only if there exists a set of divisorless μ-ideals whose set-union is the set-complement of A and which exhaust all the divisorless ideal factors of their intersection. An α-ideal $A\neq I$ is the [dual] last residue class of its last residue class C if and only if every divisorless factor of C is disjoint with A. A distributive lattice with units is a Boolean algebra if and only if every prime α-ideal is minimal-prime.

P. M. Whitman (Silver Spring, Md.).

Asano, Keizo. Über Hauptidealringe mit Kettensatz. Osaka Math. J. 1, 52–61 (1949).

Let $\mathfrak{v}$ be a ring with unity and nilpotent radical R. Call $\mathfrak{v}$ uniserial if (1) both chain conditions hold for left and right ideals, (2) $\mathfrak{v}$ is a direct sum of primary rings, and (3) every left or right ideal generated by a primitive idempotent has a unique composition series (as a $\mathfrak{v}$-module). If (3) holds for left ideals only and (1) and (2) hold, $\mathfrak{v}$ is called left-uniserial. If $\mathfrak{v}$ has a nilpotent radical R with $\mathfrak{v}/R$ semi-simple then the following conditions are equivalent: (1) $\mathfrak{v}$ is left-uniserial, (2) $\mathfrak{v}/R^2$ is left-uniserial, (3) for any two prime ideals p, q of $\mathfrak{v}$, $\mathfrak{v}/(p\cap q)^2$ is left-uniserial, (4) for any two prime ideals p, q of $\mathfrak{v}$, $pq=qp$ and $\mathfrak{v}/p^2$ is left-uniserial, (5) every left ideal of $\mathfrak{v}$ is principal, (6) every prime ideal of $\mathfrak{v}$ is a principal left ideal, (7) p/R^2 is a cyclic $\mathfrak{v}$-module for all prime ideals p in $\mathfrak{v}$, (8) $p/(p\cap q)^2$ is a cyclic $\mathfrak{v}$-module for every two prime ideals p, q of $\mathfrak{v}$, (9) if p and q are prime ideals of $\mathfrak{v}$ then $pq=qp$ and p/p^2 is a cyclic $\mathfrak{v}$-module. Analogous results hold for uniserial rings. A Frobenius ring $\mathfrak{v}$ is uniserial if for every minimal two-sided ideal m of $\mathfrak{v}$ the residue class ring of m is again a Frobenius ring. For algebras left-uniserial implies uniserial. *R. M. Thrall.*

Levitzki, Jakob. A lemma on the radical and an application. Riveon Lematematika 3, 20–24, 38 (1949). (Hebrew. English summary)

The author's summary is as follows. In the present note we use the term radical to denote the sum of all nilpotent ideals of a ring S (notation: $N(S)$ = radical of S). In § 2 the following lemma is proved: If $N(S)=0$ and A is a left ideal, then $N(A)=M(A)$, where $M(A)$ is the left annihilator of A (i.e., the set of all elements x such that $x\varepsilon A$ and $Ax=0$). By using this lemma we prove in § 3 that each right or left nilideal of a ring which satisfies a polynomial identity (in short: PI-ring) belongs to the lower radical of S. As a consequence of this theorem it follows that in each PI-ring the radicals in the sense of Koethe and in the sense of Baer exist, and coincide with each other. Hence, each PI-nil-ring coincides with its lower radical. Since the lower radical of a ring is semi-nilpotent, this consequence includes a theorem due to Kaplansky [Bull. Amer. Math. Soc. 54, 575–580 (1948); these Rev. 10, 7] who proved that each PI-nil-ring is semi-nilpotent. *I. S. Cohen* (Cambridge, Mass.).

Okugawa, Kôtaro. On the ring with derivations. Math. Japonicae 1, 152–163 (1949).

After considerable expository discussion of the Ritt theory of differential ideals the author considers a differential ideal M in the differential ring of all differential polynomials in a finite number of indeterminates over a differential field of characteristic 0. If, for any differential ideal I, $I^{(t)}$ denotes the ideal (not differential) of all differential polynomials of order not exceeding t, if $P_1, \cdots, P_r$ are the prime components of the perfect differential ideal $\{M\}$, and if $M^{(t)}=Q_{1t}\cap Q_{2t}\cap\cdots$ (a shortest primary decomposition, with Q_{it} belonging to the prime ideal P_{it}), then for t large the minimal ideals among $P_{1t}, P_{2t}, \cdots$ are precisely $P_1^{(t)}, \cdots, P_r^{(t)}$. Ordering $Q_{1t}, Q_{2t}, \cdots$ so that $P_{it}=P_i^{(t)}$ $(1\leqq i\leqq r)$, the author separates the numbers $1, \cdots, r$ into equivalence classes by defining $i\sim k$ to mean there exists a chain $P_{it}, P_{jt}, \cdots, P_{kt}$ such that each two consecutive ideals are contained in some P_{lt}. Theorem: if for all large t this separation into equivalence classes is the same then $M=M_1\cap\cdots\cap M_u$, where M_h is a differential ideal whose perfect differential ideal is the intersection of all P_i with subscript in the hth equivalence class. *E. R. Kolchin.*

Abellanas, Pedro. Orderable fields with a single automorphism. Revista Mat. Hisp.-Amer. (4) 9, 3–9 (1949). (Spanish)

The theorem proposed is that "the necessary and sufficient conditions for an ordered field K to allow only the identical automorphism are (a) that it is Archimedean and (b) that the square root of every positive element of K is also in K." The sufficiency of these conditions is established but not the necessity. [The author shows only that if either (a) or (b) does not hold he can construct an example in which the theorem fails. Condition (b) does not, in fact, hold for the field of rational numbers, so that the theorem is actually false.] *D. B. Scott* (London).

Rutherford, D. E. On commuting matrices and commutative algebras. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 454–459 (1949).

The classical theorem that a set of mutually commuting matrices (over the field of complex numbers) is simultaneously similar to a set in triangular form is proved again, with certain refinements. This result is used to give a direct

proof that every commutative algebra is a direct sum of algebras which are primary, or which become so on adjunction of a unity; these components are obtained with a suitable canonical multiplication table. The proof uses the classical canonical form for matrices and hence applies only to algebraically closed fields. *S. MacLane.*

Kochendörffer, Rudolf. Über einen speziellen Matrizenring. Math. Nachr. 1, 345–349 (1948).

The set of all h-rowed square matrices $A=(a_{ij})$ over a field K such that the row sums $R_j=\sum_i a_{ij}$ are equal for all j and such that the column sums $C_i=\sum_j a_{ij}$ are equal for all i form an algebra M over K. If the characteristic of K is zero or does not divide h then also $R_j=C_i$ and the algebra M is the direct sum of two total matrix algebras over k, one of degree 1 and the other of degree $h-1$. If the characteristic of K is a factor of h, then M has a two sided ideal N such that (a) M/N is a direct sum of two isomorphic copies of K and (b) N has radical R of order $2h-3$ such that N/R is a total matrix algebra over K of degree $h-2$.

R. M. Thrall (Ann Arbor, Mich.).

Albert, A. A. A theory of trace-admissible algebras. Proc. Nat. Acad. Sci. U. S. A. 35, 317–322 (1949).

A trace-admissible algebra is a power-associative algebra A over a field F for which there is a symmetric bilinear function $\tau(x, y)$ satisfying $\tau(x, yz)=\tau(xy, z)$, $\tau(e, e)\neq 0$ if e is idempotent, $\tau(x, y)=0$ if xy is zero or nilpotent. The radical of A is defined to be the maximal nilideal N of A, and A is called semisimple in case $N=0$. In obtaining here a structure theory for trace-admissible algebras over F of characteristic prime to thirty, the author brings together in a unified theory a number of topics previously considered by him in a paper on power-associative rings [Trans. Amer. Math. Soc. 64, 552–593 (1948); these Rev. 10, 349]. He proves that the semisimple algebra $A-N$ is trace-admissible; also that, if A is semisimple, then the attached algebra $A^{(+)}$ is a semisimple Jordan algebra, A is flexible, A is a direct sum of simple algebras A_i such that the $A_i^{(+)}$ are simple Jordan algebras. Thus, if A is a (semisimple) simple algebra of order n over a center F of characteristic 0 or $p>n$, then A is either a Jordan algebra, a flexible algebra of degree two, or a quasiassociative algebra. One limitation of the theory is that every nilalgebra (for example, every Lie algebra) is defined to be its own radical, so there are no semisimple nilalgebras. [Minor errors noted: twice in lines 28–30, p. 320, $2\tau(z, yx^2)$ should be $2\tau(z, yR_z^2)$; in line 10, p. 321, read $x(yx)-(xy)x$ for $(yx)x-x(xy)$.]

R. D. Schafer (Philadelphia, Pa.).

Koszul, Jean-Louis. Sur l'homologie et la cohomologie des algèbres de Lie. C. R. Acad. Sci. Paris 228, 288–290 (1949).

Koszul, Jean-Louis. Sur la cohomologie relative des algèbres de Lie. C. R. Acad. Sci. Paris 228, 457–459 (1949).

Let $\mathfrak{a}$ be a Lie algebra; denote by $C(\mathfrak{a})$ and $C^*(\mathfrak{a})$ the exterior algebras respectively on $\mathfrak{a}$ and on its dual space. To every X in $\mathfrak{a}$ there is associated an infinitesimal transformation $\theta(X)$ operating in $C(\mathfrak{a})$; $\theta(X)$ is the derivation of $C(\mathfrak{a})$ which extends the mapping $Y\to[X, Y]$ of $\mathfrak{a}$ into itself. An element u of $C(\mathfrak{a})$ is called invariant if $\theta(X)u=0$ for every X in $\mathfrak{a}$. If u is in $C(\mathfrak{a})$, let $\iota(u)$ be the operation of interior multiplication by u in $C^*(\mathfrak{a})$. If u is invariant, then $\iota(u)$ maps cocycles upon cocycles, and coboundaries upon coboundaries; therefore, it defines an operation $\iota'(u)$ on the cohomology algebra $H^*(\mathfrak{a})$ of $\mathfrak{a}$. A cohomology class a is called primitive if (a) a has no zero-dimensional component; (b) whenever u is an invariant element of $C(\mathfrak{a})$ which has no zero-dimensional component, $\iota'(u)a$ is a scalar.

The algebra $\mathfrak{a}$ is called reductive if the representation $X\to\theta(X)$ is semi-simple. If the basic field is of characteristic 0, this means that $\mathfrak{a}$ is the product of an Abelian and a semi-simple algebra. In that case, it is stated that the cohomology algebra $H^*(\mathfrak{a})$ of $\mathfrak{a}$ may be identified with the exterior algebra over the space of its primitive elements. This is a refinement of the theorem of Hopf [Ann. of Math. (2) 42, 22–52 (1941); these Rev. 3, 61]; moreover, a purely algebraic proof of this theorem is announced. If $\mathfrak{b}$ is a subalgebra of the Lie algebra $\mathfrak{a}$, there is a natural homomorphism π of the relative cohomology algebra $H^*(\mathfrak{a}; \mathfrak{b})$ of $\mathfrak{a}$ with respect to $\mathfrak{b}$ into $H^*(\mathfrak{a})$. Assuming that $\mathfrak{a}$ is reductive and that the basic field is of characteristic 0, the author proves theorem A: the subalgebra $\pi(H^*(\mathfrak{a}; \mathfrak{b}))$ of $H^*(\mathfrak{a})$ is generated by the unit element and a set of primitive elements. This is a generalization of a theorem of Samelson [Ann. of Math. (2) 42, 1091–1137 (1941); these Rev. 3, 143].

A subalgebra $\mathfrak{b}$ of $\mathfrak{a}$ is called a reductive subalgebra if the restriction to $\mathfrak{b}$ of the mapping $X\to\theta(X)$ is a semi-simple representation of $\mathfrak{b}$; this implies that $\mathfrak{b}$ is a reductive algebra. Assume from now on that this is the case and that the basic field is of characteristic 0. The kernel of the homomorphism π introduced above will be denoted by N^0; N^1 will denote the ideal of $H^*(\mathfrak{a}; \mathfrak{b})$ composed of the cohomology classes which may be represented by cocycles of $\mathfrak{a}$ which belong to the ideal generated by the linear functions on $\mathfrak{a}$ which vanish on $\mathfrak{b}$. The author sketches the proof of theorem B: there exists a linear mapping ρ of the space Q of primitive elements of $H^*(\mathfrak{b})$ into N^0/N^1 whose zeros are exactly the elements of Q which belong to the image space of the natural mapping of $H^*(\mathfrak{a})$ into $H^*(\mathfrak{b})$. This theorem gives for the first time an insight in the topological structure of the system formed by a compact Lie group G and a connected closed subgroup H of G which is homologous to 0 in G. Theorem B is completed by theorem D which states that, if $\rho(Q)$ contains l linearly independent homogeneous elements of degree $p+1$, then every system of homogeneous generators of $H^*(\mathfrak{a}; \mathfrak{b})$ contains at least l elements of degree $p+1$. Theorem C relates the situation described above to the Leray theory of the homology ring of a mapping.

C. Chevalley (New York, N. Y.).

Gotô, Morikuni. On a theorem of E. E. Levi. Math. Japonicae 1, 104–106 (1948).

L'auteur donne une nouvelle démonstration du théorème suivant: soit G une algèbre de Lie sur le corps complexe, dont le radical est R; alors on a une décomposition $G=R\oplus S$ où S est une sous-algèbre semi-simple, déterminée à un automorphisme intérieur près. La démonstration procède comme suit: on se ramène au cas où R est commutatif et où le centre de G est nul; puis on introduit le groupe adjoint $\mathfrak{G}$ engendré par intégration de la représentation régulière gauche de G (laquelle est fidèle en vertu des hypothèses faites); à R correspond alors un sous-groupe fermé $\mathfrak{R}$ de $\mathfrak{G}$, isomorphe à un espace vectoriel, et tel que $\mathfrak{G}^*=\mathfrak{G}/\mathfrak{R}$ soit semi-simple complexe; soit alors $\mathfrak{H}^*$ un sous-groupe compact maximal de $\mathfrak{G}^*$, dont l'image inverse dans $\mathfrak{G}$ est un sous-groupe $\mathfrak{H}$ contenant $\mathfrak{R}$; d'après un résultat de Iwasawa [Ann. of Math. (2) 50, 507–558 (1949), lemma 3.7, p. 521; ces Rev. 10, 679] il existe un sous-groupe compact $\mathfrak{K}$ de $\mathfrak{H}$ tel que $\mathfrak{H}=\mathfrak{K}\cdot\mathfrak{R}$, avec $\mathfrak{K}\cap\mathfrak{R}=\{e\}$; si n est la dimension

complexe de $\mathfrak{G}^*$, on peut alors associer à $\mathfrak{R}$ une sous-algèbre (réelle) K de G de dimension (réelle) n, et en repassant du réel au complexe celle-ci engendre la sous-algèbre S cherchée; le fait que S soit déterminée à un automorphisme intérieur près résulte de la propriété analogue établie par Iwasawa pour le sous-groupe $\mathfrak{R}$. *R. Godement* (Nancy).

THEORY OF GROUPS

Nakayama, Tadasi. Note on faithful modular representations of a finite group. J. Math. Soc. Japan 1, 10–14 (1948).

If a group G possesses a faithful nonmodular irreducible representation $M(G)$, then any directly indecomposable component of a modular regular representation of G is faithful if it contains as a constituent a modular representation induced by $M(G)$; moreover, every faithful directly indecomposable constituent of the modular regular representation of G will contain as a constituent a modular representation induced by some faithful nonmodular irreducible representation. These theorems extend results obtained by the author [forthcoming paper], and also obtained independently at about the same time by R. Kochendörffer [Math. Nachr. 1, 25–39 (1948); these Rev. 10, 281]. *R. M. Thrall* (Ann Arbor, Mich.).

*Prokop, Wilfried. **Über eine Formel von Frobenius zur Berechnung der Charaktere endlicher Gruppen.** Thesis, Eidgenössische Technische Hochschule, Zürich, 1948. 39 pp.

Let $H_1, \cdots, H_s$ be (proper) subgroups of a finite group G. The author asks under what circumstances is every character of G a linear combination with integral coefficients of the characters of G induced by the characters of all of the H_i. Two necessary conditions are (1) that for every class of G there shall be at least one of the subgroups H_i which contains an element of that class; and (2) that the greater common divisor of the indices in G of the groups H_i shall be unity. Condition (1) never holds in a cyclic group and condition (2) never holds in a p-group. Certain determinantal conditions are given for determining when any set of characters of G can form an integral generating system for all characters of G. *R. M. Thrall.*

Ribeiro, Hugo. "Lattices" des groupes abéliens finis. Comment. Math. Helv. 23, 1–17 (1949).

Lattices of (all subgroups of) finite Abelian groups are studied [cf. R. Baer, Amer. J. Math. 61, 1–44 (1939); K. Iwasawa, J. Fac. Sci. Imp. Univ. Tokyo. Sect. I. 4, 171–199 (1941); these Rev. 3, 193; A. W. Jones, Duke Math. J. 12, 541–560 (1945); these Rev. 7, 114; G. Zappa, Pont. Acad. Sci. Comment. 8, 443–460 (1944); these Rev. 10, 352; G. Birkhoff, Lattice Theory, revised ed., Amer. Math. Soc. Colloquium Publ., v. 25, New York, 1948, pp. 96–99; these Rev. 10, 673]. Such a lattice is uniquely factorable as the product of the lattices of the primary components (subgroups consisting of the elements whose orders are powers of a given prime). A finite modular lattice is "exceptional" if some convex sublattice is a projective geometry not over a field. A finite nonexceptional modular lattice is "u-uniform" if every complemented convex sublattice of dimension 2 has $u+3$ elements. A finite lattice is 1-uniform if and only if it is distributive; a modular lattice is distributive if all its convex 2-dimensional sublattices are distributive.

The lattices of direct products of cyclic groups all of order p^n are those which are p-uniform and in which each irreducible element is a cycle (all lesser elements also irreducible), each maximal cycle has dimension α, and the number of elements covering a given element does not exceed the number of atoms. Such a lattice is "the extension of degree α of a projective geometry over the field of order p." A lattice is isomorphic to the lattice of a primary component if and only if it is an ideal of an extension of a projective geometry over a finite field. If a primary component is the direct product of m cyclic groups of which m_i have order p^{α_i} ($\sum m_i = m$), $\alpha_1 > \alpha_2 > \cdots > \alpha_l$, then the field has order p, the projective geometry has dimension m, the extension has degree at least α_1, and there exist cycles, m_i of which have dimension α_i, which are independent and whose union is the ideal. A lattice is the lattice of a finite Abelian group if and only if it is the product of ideals of extensions of u-uniform complemented lattices, the u being prime and distinct; the representation as such a product is unique, the factor (if any) with $u=1$ being the lattice of a cyclic group. The lattice of a finite Abelian group determines the group if and only if $u \neq 1$ in each factor of the above representation (no factor is distributive). *P. M. Whitman.*

Taussky, Olga, and Todd, John. Covering theorems for groups. Ann. Soc. Polon. Math. 21 (1948), 303–305 (1949).

Let G be an Abelian group having n basis elements each of the order d. Let S denote the set of the $n(d-1)+1$ distinct powers of the basis elements. The authors set the problem: what is the minimal integer $\sigma = \sigma(n, d)$ so that there exists a subset H of G having σ elements, such that $G = H \cdot S$? They show that $\sigma(4, 3) = 9$, $\sigma(7, 2) = 16$. The general case seems to be difficult. *P. Erdös.*

Szélpál, I. Die Abelschen Gruppen ohne eigentliche Homomorphismen. Acta Univ. Szeged. Sect. Sci. Math. 13, 51–53 (1949).

If G is an Abelian group for which every homomorphic image $\bar{G} \neq 0$ is an isomorphic image, then G is either a cyclic group of order p or the group with generators $A_1, A_2, \cdots$ such that $A_1 \neq 0$, $pA_1 = 0$, $pA_2 = A_1, \cdots, pA_n = A_{n-1}, \cdots$, where p is a prime. *R. M. Thrall* (Ann Arbor, Mich.).

Szele, T. Die Abelschen Gruppen ohne eigentliche Endomorphismen. Acta Univ. Szeged. Sect. Sci. Math. 13, 54–56 (1949).

If G is a group for which every nonzero endomorphism is an automorphism then either G is the additive group of all rational numbers or a cyclic group of prime order p. *R. M. Thrall* (Ann Arbor, Mich.).

Takahasi, Mutuo. On partitions of free products of groups. Osaka Math. J. 1, 49–51 (1949).

A partition of a group G is a set of subgroups such that every element of G except the identity is contained in exactly one of the subgroups. It is shown here that a free product $A*B$ has a partition which consists of all conjugates of A, all conjugates of B and a set of infinite cyclic groups. As noted at the end of the paper, this result has also been obtained by P. G. Kontorovič [Mat. Sbornik N.S. 22(64), 79–100 (1948); these Rev. 9, 493]. *M. Hall.*

Neumann, Hanna. Generalized free products with amalgamated subgroups. II. Amer. J. Math. 71, 491–540 (1949).

In part I [same J. 70, 590–625 (1948); these Rev. 10, 233] the generalized free product G of groups G_α with amalgamated subgroups $U_{\alpha\beta}$ was defined. The product G contains as subgroup a group U which is itself a generalized free product of subgroups U_α of the G_α with the same amalgamated subgroups $U_{\alpha\beta}$; and no subgroup of U has this property. The main result of the present paper is that every subgroup H of G which is not contained in U is itself a generalized free product. The amalgamated subgroups of the factors of H are all contained in conjugates of U. The factors of H are of two kinds: (1) subgroups of conjugates of the factors G_α of G and (2) factors F not of the first kind. The amalgamated subgroups of such an F generate a normal subgroup F' with F/F' an infinite cyclic group. The proofs of these results are quite complex, and are based primarily on a concept of length for elements of G. Generators for the subgroup H are chosen according to a complicated induction procedure and in such a way as to exhibit the factors of H. The process fails to give any information if H is a subgroup of U. *R. M. Thrall* (Ann Arbor, Mich.).

Eilenberg, Samuel. Topological methods in abstract algebra. Cohomology theory of groups. Bull. Amer. Math. Soc. 55, 3–37 (1949).

In den letzten Jahren sind in der abstrakten Algebra der Gruppen, Ringe usw. Begriffe und Methoden entwickelt worden, die zwar rein algebraisch sind, aber aus der algebraischen Topologie stammen, insofern als sie dort erstmalig und in natürlicher Weise auftraten oder durch die dort üblichen Prozesse nahegelegt wurden. Einer beliebigen Gruppe, einer Algebra oder einem Lie-Ring werden ähnlich wie einem Komplex in der Topologie Abelsche Gruppen H^q $(q=0, 1, 2, \cdots)$, Cohomologiegruppen genannt, zugeordnet; diese Gruppen bringen interessante Eigenschaften der betrachteten algebraischen Struktur zum Ausdruck und stehen in mannigfachen Beziehungen zu andern algebraischen Theorien. Im Falle der Cohomologiegruppen von Gruppen (auch von Lie-Ringen) gestatten die rein algebraischen Begriffe und Ergebnisse ihrerseits Anwendungen auf topologische Fragen. Die vorliegende vorzügliche Übersicht über diesen Gegenstand ist aus einem 1945 gehaltenen Vortrage entstanden, trägt aber auch der gesamten späteren Entwicklung bis 1948 Rechnung und enthält in einzelnen Punkten neue Resultate und Darstellungen; sie beschränkt sich im Wesentlichen auf den Fall von Gruppen.

Die ersten zwei Abschnitte enthalten die Definitionen samt einigen Varianten und Vereinfachungen. In der Definition der Cohomologiegruppen $H^q=H^q(Q, G)$ einer beliebigen Gruppe Q tritt neben Q noch eine beliebig vorgegebene Abelsche Gruppe G auf, welche Q als Operatorgruppe besitzt. In § 3 wird die algebraische Rolle von H^0 und H^1 angegeben: H^1 hängt zusammen mit einem Spezialfall des Erweiterungsproblems für Gruppen mit Operatoren. Wie in den §§ 4–6 ausgeführt wird, spielen H^2 und H^3 eine Rolle für die Erweiterungstheorie der Gruppen (im Sinne von Schreier). Die Gruppe $H^2(Q, G)$ beschreibt einerseits die Gesamtheit der Erweiterungen von Q durch G, die zu den vorgegeben Operationen von Q in G gehören; andererseits die Erweiterungen von Q durch beliebige Gruppen K mit dem Zentrum G (für eine vorgegeben Darstellung von Q durch äussere Automorphismen von K, K wird in diesem Fall ein Q-Kern genannt), falls es solche Erweiterungen überhaupt gibt: die Q-Kerne mit dem Zentrum G lassen sich nämlich den Elementen von $H^3(Q, G)$ zuordnen, und das Erweiterungsproblem ist nur lösbar für Kerne K, die zu $0 \in H^3(Q, G)$ gehören. Für die weitern H^q gelten ähnliche Resultate wie für H^3, insofern als sie in gewissen Fällen die Erweiterungsmöglichkeiten einer Gruppe zu einem nichtassoziativen System beschreiben können (§ 7). § 8 ist den Beziehungen zur Galoisschen Theorie gewidmet. Hier ist Q die Galoissche Gruppe einer Körpererweiterung $N \supset P$ und G die multiplikative Gruppe von N (die additive ergibt nur triviale Gruppen H^q). Die Gruppen H^2 und H^3 stehen in engem Zusammenhang mit den Gruppen der Algebrenklassen über P und N. In den folgenden Abschnitten werden verschiedene Eigenschaften der Gruppen $H^q(Q, G)$ diskutiert; es gelten insbesondere zwei Reduktionssätze, die zwischen den H^q für verschiedene q (und für geeignete Gruppen G) einfache Beziehungen herstellen. Für zyklische Gruppen Q werden die H^q in neuer Weise berechnet. Nach dem Muster der algebraischen Topologie werden den Cohomologiegruppen $H^q(Q, G)$ Homologiegruppen zur Seite gestellt; interessante Ausdrücke ergeben sich speziell, wenn Q durch Erzeugende und Relationen (d.h. als Faktorgruppe einer freien Gruppe) dargestellt ist.

Der restliche Teil der Ausführungen (§§ 14–17) gehört der Topologie an. Die genannten algebraischen Bildungen sind geeignet (und auch zuerst gerade zu diesem Zwecke eingeführt worden), um Zusammenhänge zwischen der Fundamentalgruppe und den Homologieeigenschaften gewisser Räume zu beschreiben. Im einfachsten Falle handelt es sich um einen asphärischen Raum R mit der Fundamentalgruppe Q; für alle q ist dann die q-te Cohomologiegruppe von R mit Koeffizienten aus G isomorph zu $H^q(Q, G)$ (mit trivialen Operationen von Q in G). [Die Existenz derartiger Beziehungen, wenn auch nicht ihre explizite algebraische Form, ist erstmals in den Arbeiten von Hurewicz über die Theorie der Deformationen festgestellt worden, die wohl hier Erwähnung verdient hätten.] Einen andern Ausdruck für denselben Sachverhalt erhält man, wenn man von einem in allen Dimensionen azyklischen Raum ausgeht, in welchem eine Automorphismengruppe Q operiert. Diese beiden Formulierungen bilden den Ausgangspunkt für eine Reihe verschiedener Verschärfungen und Verallgemeinerungen, sowohl der topologischen Resultate wie auch der algebraischen Begriffsbildungen. Besonders erwähnt sei die neue topologische Invariante k^{n+1} eines Raumes R, dessen Homotopiegruppen π_q für $1<q<n$ trivial sind; k^{n+1} ist ein Element von $H^{n+1}(\pi_1, \pi_n)$, wobei π_1 in natürlicher Weise in π_n operiert. Diese Invariante kann als "Hindernis" bei der Erweiterung einer gewissen Abbildung aufgefasst werden; überhaupt bestehen im algebraischen und im topologischen Teil starke Analogien und Beziehungen zur Theorie der Hindernisse (in der Topologie der Abbildungen und Faserungen), auf die wiederholt hingewiesen wird. Ein ausführliches Literaturverzeichnis orientiert über die Originalarbeiten. Die verschiedenen Grundbegriffe und die topologische Theorie stammen von Hopf, Eilenberg-MacLane, Freudenthal und Eckmann; die algebraische Cohomologietheorie der Algebren wurde von Hochschild entwickelt, diejenige der Gruppen mit allen ihren Anwendungen von Eilenberg und MacLane. *B. Eckmann* (Zürich).

Lyndon, Roger. New proof for a theorem of Eilenberg and MacLane. Ann. of Math. (2) 50, 731–735 (1949).

The theorem in question gives a method of computing the $(n+2)$th cohomology group $H^{n+2}(G, K)$ of a group G oper-

ating on the right on an Abelian coefficient group K, in terms of any given representation $G=F/R$ of G by a free group F. It asserts that $H^{n+1}(G, K)$ for $n>0$ is isomorphic to $H^n(G, \mathrm{Hom}(R, K))$, where G operates both on the right and on the left on the new coefficient group $\mathrm{Hom}(R, K)$ of all homomorphisms h of R into K, according to the rules $[h^g](r)=h(r)^g$ and $[{}^g h](r)=h(a^{-1}ra)$, where $g\varepsilon G$, $r\varepsilon R$ and a is any element of the free group F in the coset g of $G=F/R$. The author's new proof operates directly with generators and words of the free group F, using explicitly the fact that any n-cocycle ($n>1$) of a free group is a coboundary; it is thus shorter than the original inductive proof of the theorem [same Ann. (2) **48**, 51–78 (1947); these Rev. **8**, 367].

S. MacLane (Chicago, Ill.).

Tamari, Dov. Groupoïdes ordonnés. L'ordre lexicographique pondéré. C. R. Acad. Sci. Paris **228**, 1909–1911 (1949).

Let $\mathfrak{D}$ be a semi-group, regular on the right in the sense of Ore, and partially ordered on the right in the sense defined by the author in a previous paper [same vol., 1184–1186 (1949); these Rev. **10**, 508]. Necessary and sufficient conditions that the order relation in $\mathfrak{D}$ can be extended to the quotient group $Q(\mathfrak{D})$ of $\mathfrak{D}$ are the right cancellation laws: if $ac<bc$ and c is a right conserver, then $a<b$; if $ar<br$ and r is a right inverter, then $a>b$. A similar theorem holds for rings partially ordered on the right. If a right partially ordered ring A contains a right multiplicative conserver a such that a and $-a$ are comparable, then $-a$ is a right multiplicative inverter. If A contains two right nondivisors of zero whose sum is also a right nondivisor of zero, then the additive group of A is conservative. Returning to semigroups, let $\mathfrak{L}$ be the free semi-group on a simply ordered set of generators. The lexicographic ordering Ω_λ of $\mathfrak{L}$ defined as in a dictionary is admissible on the left but not on the right. Another simple ordering Ω_l of $\mathfrak{L}$ is defined as follows: $p<q$ if the word p of $\mathfrak{L}$ is shorter than the word q, or if they have the same length and p precedes q lexicographically. The ordering Ω_l is admissible on both sides and is moreover Archimedean. The weight of q with respect to p is defined to be ∞ if $p^\nu<q$ for every positive integer ν, and otherwise to be $\lim \nu_n/n$, where ν_n is defined by $p^{\nu_n}\leqq q^n<p^{\nu_n+1}$. The author states that Ω_λ and Ω_l are extremal cases of the weighted order Ω_ν, but does not define the latter.

A. H. Clifford (Baltimore, Md.).

Fuchs, Ladislas. Absolutes in partially ordered groups. Nederl. Akad. Wetensch., Proc. **52**, 251–255 = Indagationes Math. **11**, 66–70 (1949).

The absolute $|a|$ of an element a of a lattice-ordered group G can be defined by $|a|=a\cup -a$ or equivalently by $|a|=a^+-a^-$, where $a^+=a\cup 0$ and $a^-=a\cap 0$ [L. V. Kantorovitch, Rec. Math. [Mat. Sbornik] N.S. **2**(**44**), 121–165 (1937); G. Birkhoff, Ann. of Math. (2) **43**, 298–331 (1942); these Rev. **4**, 3]. This is extended (in two ways) to any partially ordered group G by defining $|a|$ to be a set of elements of G, rather than a single element. Denote by $U(a_1, \cdots, a_n)=U(a_i)$ the set of all x in G such that $x\geqq a_i$ ($i=1, 2, \cdots, n$), and define $L(a_1, \cdots, a_n)$ dually. Then $|a|$ is defined to be the set a^+-a^-, where $a^+=U(a, 0)$ and $a^-=L(a, 0)$; $\|a\|$ is defined to be $U(a, -a)$. We have $|a|\subseteq\|a\|$, and equality holds if G satisfies the following two conditions: $U(a_i)+U(b_j)=U(a_i+b_j)$; $na\geqq 0$ implies $a\geqq 0$. The following are shown: (1) $|0|=G^+$; $|a|\subset G^+$ if $a\neq 0$; (2) $|-a|=|a|$; (3) $|a-b|=U(a,b)-L(a,b)$; (4) $n|a|\subseteq|na|$; (5) $|a+b|\supseteq|a|+|b|$ provided G is commutative. Relations (2), (4), and (5) hold for $\|a\|$ also.

A. H. Clifford (Baltimore, Md.).

Ellis, David. Superposability properties of naturally metrized groups. Bull. Amer. Math. Soc. **55**, 639–640 (1949).

Let G represent an Abelian group written additively. Menger defined for any pair of elements p and q of G a predistance (Abstand) pq as the unordered pair of the differences $p-q$ and $q-p$. Two subsets of G are then congruent if there exists a predistance-preserving mapping of one onto the other (called congruence), superposable if the mapping may be extended into a congruence of G onto itself. In this note it is proved that these notions are equivalent or in other words that G has the property of free mobility. Actually any congruence may be extended into a transformation of the group Γ generated by the translations $x\to x+t$ and the symmetry $x=-x$ in G. *C. Y. Pauc.*

Tits, J. Généralisations des groupes projectifs. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 197–208 (1949).

Let E be an abstract set of at least three elements. A group of one-to-one correspondences of E with itself (proper transformations) is triply transitive provided to each two triples p, q, r and p', q', r' of pairwise distinct elements of E there exists one and only one correspondence of the group such that p, q, r correspond, respectively, to p', q', r'. If p, q are any two elements of E, a degenerate transformation (p, q) of E is one that sends p into all the points of E, and all the points of E into q. A set G of transformations obtained by adjoining to a group of triply transitive correspondences all the degenerate transformations of E is triply transitive in the wide sense. The author derives properties of this generalization of (one-dimensional) projective transformations. He shows that in terms of a suitably defined formalism every proper transformation of G can be expressed in one of the two forms $y=ax+b$, $y=a/(x+c)+b$.

L. M. Blumenthal (Columbia, Mo.).

Tits, J. Généralisations des groupes projectifs. II. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 224–233 (1949).

In the paper reviewed above the author defined and studied triply transitive sets of transformations of an abstract space E into itself. Such sets generalize the group of one-dimensional projective transformations. In this note projective groups are characterized among triply transitive sets of transformations, and theorems of the following kind are obtained. In order that a triply transitive group may be projective it is necessary and sufficient that any two pairs of elements determine an involution.

L. M. Blumenthal (Columbia, Mo.).

Calabi, L., et Ehresmann, Charles. Sur les extensions de groupes topologiques. C. R. Acad. Sci. Paris **228**, 1551–1553 (1949).

The authors formulate the concept of topological group extension: it is a group $E(B, F)$ with F as normal subgroup and $E/F\approx B$; the extension is called fibered if the canonical map $E\to B$ is a fiber map with F as structure group. An n-extension is defined by an iterative procedure in the obvious way. Several theorems of the following type are stated. Every Abelian topological extension of R^n (additive group of n-space) by a locally compact Abelian group is trivial. It is pointed out that every fibered extension of one toral group T^ν by another T^n is trivial, whereas there exist

nontrivial fiber bundles with T^n as fiber, T^s as base, and translations as structure group. *H. Samelson.*

Nagao, Hirosi. The extension of topological groups. Osaka Math. J. 1, 36–42 (1949).

Let $\mathfrak{N}$, $\mathfrak{A}$ be groups. An extension of $\mathfrak{N}$ by $\mathfrak{A}$ is formed by a group $\mathfrak{G}$, an isomorphism into $i:\mathfrak{N}\to\mathfrak{G}$, a homomorphism onto $\pi:\mathfrak{G}\to\mathfrak{A}$ with $i(\mathfrak{N})=\pi^{-1}(e)$. A representative is a map $S:\mathfrak{A}\to\mathfrak{G}$ such that $\pi S_t=t$, $S_{t^{-1}}=S_t^{-1}$. Define the maps $C:\mathfrak{A}\times\mathfrak{A}\to\mathfrak{N}$, $T:\mathfrak{A}\to\text{aut }\mathfrak{N}=$group of automorphisms, by $C_{x,y}=i^{-1}(S_xS_yS_{xy}^{-1})$ and $T_x(u)=i^{-1}\{S_xi(u)S_x^{-1}\}$. As shown by O. Schreier [Monatsh. Math. Phys. 34, 165–180 (1926)], we have:

$$T_xT_y(u)=C_{x,y}T_{xy}(u)C_{x,y}^{-1}, \tag{1}$$

$$C_{x,y}C_{xy,z}=T_x(C_{y,z})C_{x,yz}, \tag{2}$$

$$C_{x,x^{-1}}=e, \tag{3}$$

and conversely, if (1), (2), (3) hold for given $\mathfrak{N}$, $\mathfrak{A}$, C, T, the set $\mathfrak{G}=\mathfrak{N}\times\mathfrak{A}$ which is a group under the product $(u,x)(v,y)=(uT_x(v)C_{x,y},xy)$, the map i defined by $i(u)=(u,e)$ and the map π defined by $\pi(u,x)=x$ form an extension of $\mathfrak{N}$ by $\mathfrak{A}$; moreover the map S defined by $S(x)=(e,x)$ is a representative inducing C, T. If $\mathfrak{N}$, $\mathfrak{A}$, $\mathfrak{G}$, i are topological and π is continuous and open, we have a topological extension. In this paper the author shows that, if S is continuous at e, then we also have: (4) the map $(x,y)\to C_{x,y}$ is continuous at (e,e), (5) the map $x\to C_{a,x}C_{ax,a^{-1}}$ is continuous at e for every a, (6) the map $(u,x)\to T_x(u)$ is continuous at each (n,e). Conversely, if (1), $\cdots$, (6) hold for given $\mathfrak{N}$, $\mathfrak{A}$, C, T, where $\mathfrak{N}$, $\mathfrak{A}$ are topological and aut $\mathfrak{N}$ refers to the homeomorphic automorphisms, the Schreier extension can be made in a unique way into a topological extension such that S is continuous at e: the products $V\times W$ of neighborhoods of e in $\mathfrak{N}$ and $\mathfrak{A}$ form a base at the identity (e,e) of $\mathfrak{G}$ (but the topology of $\mathfrak{G}$ will not be the product topology, for $\mathfrak{G}$ is not the product group). The author observes that, for a topological extension satisfying the first countability axiom, a representative continuous at e always exists. He also proves that two distinct topologies satisfying this axiom and making an extension of two topological groups into a topological extension are incomparable, i.e., neither has fewer open sets than the other. Restricting attention to locally compact Abelian groups and extensions, he finally remarks that there is a natural one-to-one map between extensions of $\mathfrak{N}$ by $\mathfrak{A}$ and those of the dual of $\mathfrak{A}$ by the dual of $\mathfrak{N}$ which preserves the Baer product of two extensions. *L. Nachbin* (Chicago, Ill.).

Mackey, George W. A theorem of Stone and von Neumann. Duke Math. J. 16, 313–326 (1949).

Soient G un groupe localement compact séparable, L^2 l'espace de Hilbert construit sur la mesure invariante à droite, et considérons dans cet espace les opérateurs suivants: $U_sf(x)=f(xs)$; $V_hf(x)=h(x)f(x)$ (h est une fonction borélienne bornée arbitraire). Ces opérateurs vérifient les relations $U_sV_h=V_{h_s}U_s$ où $h_s(x)=h(xs)$; V_h est en outre une fonction "faiblement continue" de h; enfin, ce système d'opérateurs est évidemment irréductible. L'auteur démontre que, à des équivalences unitaires et à des formations de sommes directes près, ce système est caractérisé par les relations de commutation ci-dessus. Dans le cas où G est le groupe additif R^n, on peut engendrer le système d'opérateurs en question au moyen des opérateurs autoadjoints suivants: $P_k=i\partial/\partial x_k$; $Q_k=x_k$; le théorème général redonne alors le résultat bien connu de M. H. Stone et J. von Neumann et relatif à l'"unicité" des opérateurs de Schrödinger. L'auteur indique également que ses méthodes permettent d'étudier une classe de groupes comprenant le groupe $ax+b$ de la droite. *R. Godement* (Nancy).

Montgomery, Deane. Theorems on the topological structure of locally compact groups. Ann. of Math. (2) 50, 570–580 (1949).

Extending previous results [Amer. J. Math. 70, 327–332 (1948); same Ann. (2) 49, 373–378 (1948); these Rev. 10, 591; 9, 566], the author proves several theorems on the local topological structure of an n-dimensional locally compact topological group, of which we cite some. There exists a neighborhood U of e, such that no compact subset of U carries a nonbounding n-cycle (mod 1). If A is a closed n-dimensional subset, then there are a point a in A and a neighborhood V of a, such that A contains the a-component of $\bar{V}$; if the group is locally connected, then an n-dimensional closed subset must contain inner points (well known for E^n). If U is any neighborhood of e, it contains a neighborhood V of e such that every $(n-1)$-cycle in $\bar{V}$ bounds in $\bar{U}$. If H is a closed subgroup of dimension n (of dimension $n-1$, with dimension G/H finite) then G/H is of dimension 0 (dimension 1). Some of the theorems on groups (e.g., the inner point property) are extended to homogeneous spaces G/H. All the theorems support the current conjectures on the structure of locally compact groups. *H. Samelson.*

Yoshizawa, Hisaaki. Unitary representations of locally compact groups. Reproduction of Gelfand-Raikov's theorem. Osaka Math. J. 1, 81–89 (1949).

This paper is essentially the same, both in content and in method, as that of Gelfand and Raikov [Rec. Math. [Mat. Sbornik] N.S. 13(55), 301–316 (1943); these Rev. 6, 147], which was not available to the author, although its general character was known to him. *I. E. Segal* (Chicago, Ill.).

Yoshizawa, Hisaaki. On some types of convergence of positive definite functions. Osaka Math. J. 1, 90–94 (1949).

Démonstration du théorème suivant: soient G un groupe localement compact et P_1 l'ensemble des fonctions continues de type positif prenant la valeur un pour $x=e$; alors la topologie faible sur P_1 (définie en considérant P_1 comme une ensemble de formes linéaires sur l'espace L^1) est identique à celle de la convergence uniforme sur toute partie compacte de G. [Comme l'auteur le signale, ce résultat est dû à D. A. Raĭkov [Doklady Akad. Nauk SSSR (N.S.) 58, 1279–1282 (1947); ces Rev. 9, 328]; en outre, sa démonstration est à peu de choses près identique à celle de Raikov, dont il n'a pas eu connaissance.] *R. Godement.*

Leray, Jean. Application continue commutant avec les éléments d'un groupe de Lie compact. C. R. Acad. Sci. Paris 228, 1784–1786 (1949).

Denote by H_Z the cohomology ring of Z over a field of characteristic zero. Let X, Y be locally compact and let ξ be a mapping $X\to Y$. The invariants of ξ defined by the author in an earlier paper [same C.R. 222, 1366–1368, 1419–1422 (1946); these Rev. 8, 49] include a certain system H_r of differential algebras $2\leqq r\leqq\infty$. Here H_2 is the tensor product $H_Y\otimes H_F$ when ξ is the projection of a fibered space X with simply connected fiber F onto the base Y or if ξ is the natural mapping of a group X onto a homogeneous space X/F. Let G be a compact connected Lie group. The author has associated with G [same vol., 1545–1547 (1949); these

Rev. 10, 680] a graded vector space Δ of the same rank as G. The elements of Δ are systems $\delta=\{\delta_Z\}$, where δ_Z is a differential of H_Z, Z being any space in which G operates. Suppose now that G operates on X and Y in such a way that the elements of G permute with ξ. The definition of δ can then be extended in such a way that δ becomes a differential of each algebra H_r. The author considers in particular the case in which $X=G$ and ξ is the projection of X onto a homogeneous space $Y=X/F$. An analysis of the interrelations involved here allows the author to complement his earlier results [see the preceding reference].

P. A. Smith (New York, N. Y.).

NUMBER THEORY

Storchi, Edoardo. Uguaglianze fra somme di biquadrati. Boll. Un. Mat. Ital. (3) **3**, 220–223 (1948).

Identities are given for yielding integral solutions of the equations $\sum_{i=1}^{n}x_i^4=\sum_{i=1}^{n}y_i^4$ for $n=3$, 4 and 5. An identity is also given for obtaining solutions of the problem of finding two fourth powers whose difference is the sum of two sixth powers. With slight modification this identity can be used to obtain integral solutions of the equation $x^4=y^4+z^4+t^{4n}$ when n is odd. *W. H. Simons.*

Palamà, Giuseppe. Contributo dei recenti risultati delle multigrade al problema di Waring. Boll. Un. Mat. Ital. (3) **4**, 75–79 (1949).

E. M. Wright published results [J. London Math. Soc. **9**, 267–272 (1934); **10**, 94–99 (1935); Quart. J. Math., Oxford Ser. **7**, 43–45 (1936)] concerning orders and bounds of functions connected with Waring's problem. Combining these results with data from A. Gloden's Mehrgradige Gleichungen [Noordhoff, Groningen, 1944; these Rev. **8**, 441] the author derives some improvements.

N. G. W. H. Beeger (Amsterdam).

Claudian, Virgil. Identities and Diophantine analysis. Gaz. Mat., Bucureşti **54**, 292–309 (1949). (Romanian)

Vandiver, H. S. On congruences which relate the Fermat and Wilson quotients to the Bernoulli numbers. Proc. Nat. Acad. Sci. U. S. A. **35**, 332–337 (1949).

As typical results the following may be quoted:

$$2W\equiv-\sum_{m=1}^{p-1}\lambda(m,1)\pmod p,$$

$$q(r)+2W\equiv-\sum_{m=1}^{p-1}\frac{\lambda(m,r)}{r}\pmod p,$$

where $0<r<p$ and

$$W=\frac{(p-1)!+1}{p},\quad q(r)=\frac{r^{p-1}-1}{p},\quad \lambda(m,n)=\sum{}'\frac{b^r-1}{b^p-1},$$

$\sum'$ means summation over all mth roots of unity other than 1. *L. Carlitz* (Durham, N. C.).

Steuerwald, Rudolf. Über die Kongruenz $2^{n-1}\equiv1 \pmod n$. S.-B. Math.-Nat. Kl. Bayer. Akad. Wiss. **1947**, 177 (1949).

The author gives a brief proof of the known result that there exist infinitely many nonprime n for which $2^{n-1}\equiv1 \pmod n$, by showing that $2^{n_0-1}\equiv1 \pmod{n_0}$, $n_1=2^{n_0}-1$, imply $2^{n_1-1}\equiv1 \pmod{n_1}$. *R. Bellman.*

Ferrier, A. Note on the factors of 2^n+1. Math. Tables and Other Aids to Computation **3**, 496–497 (1949).

The author proves that $(2^{92}+1)/17$ is a prime, in fact the fifth largest known. The numbers $(2^{67}+1)/3$ and $(2^{71}+1)/3$ are proved composite, the former being equal to 7327657·671 31031 82899. *D. H. Lehmer* (Berkeley, Calif.).

Uhler, Horace Scudder. The Arabian Nights' factorial and the weighted-mean factorial. Scripta Math. **15**, 94–96 (1949).

The number 450!, when written in the decimal system, has precisely 1001 digits. The author presents these digits with the exception of the last 111 which are all zero. The value of $448!/10^{109}$ is likewise given and checked by Wilson's theorem. The author notes that the formula $\sum_{k=1}^{n}k\cdot k!=(n+1)!-1$ suggests a weighted sum of the first n factorials. The average value, with this weighting, of these factorials is almost $2(n-1)!$. *D. H. Lehmer.*

Yakovkin, M. V. On a theorem of Pólya. Doklady Akad. Nauk SSSR (N.S.) **66**, 169–172 (1949). (Russian)

Let $f(x)$ be a polynomial of degree n with integral coefficients and let R_U and R_L be respectively the upper and lower bounds of the real parts of the roots of $f(x)$. The author proves two theorems. (I) If integers k_1 and k_2 exist satisfying the conditions (a) $k_1\geqq R_U-\frac{1}{2}k_2$, $f(k_1)\neq0$, $f(k_1+k_2)$ prime, or (b) $k_1\leqq R_L-\frac{1}{2}k_2$, $f(k_1+k_2)\neq0$, $f(k_1)$ prime, then $f(x)$ is irreducible in the field of the rational numbers. (II) If $f(x)$ possesses a divisor $\varphi(x)$ of degree m in the field of the rational numbers and if $|q_1^n f(p_1/q_1)|>P^2$ (P a nonnegative integer) then the integer $\{q_1q_2/(q_1,q_2)\}^n f(p_1/q_1\pm p_2/q_2)$, where the + sign is taken when $p_1/q_1\geqq R_U-\frac{1}{2}p_2/q_2$, and the − sign when $p_1/q_1\leqq R_L+\frac{1}{2}p_2/q_2$, must possess at least one divisor of modulus exceeding $(P+1)|q_2/(q_1,q_2)|^m$ or $(P+1)|q_2/(q_1,q_2)|^{n-m}$. Result (II) is a generalisation of a theorem of Pólya ($P=0$, $p_1=k-1$, $p_2=q_1=q_2=1$) [Pólya and Szegö, Aufgaben und Lehrsätze, v. 2, Springer, Berlin, 1929, p. 137, problem 127]. *R. A. Rankin.*

Ankeny, N. C. A note on the minimum of a binary form. Bull. Amer. Math. Soc. **55**, 615–618 (1949).

Let D be the absolute value of the discriminant of a binary n-ic form f with real coefficients, and let $L(f)$ denote the lower bound of $|f|$ for integral values of its variables not both zero. If s denotes the number of pairs of conjugate imaginary roots, it is known, for $n=2$ and 3, that $L(f)\leqq cD$, where c is a constant depending only on s and n. Thus, $c=\frac{1}{2}$ if $s=1$, $n=2$; $c=1/23$ if $s=1$, $n=3$. By demonstrating the existence of forms with D near zero but L bounded away from zero, it is shown here that no such inequality holds if $n>3$. The proof develops properties of continued fractions such as the following. (i) If p_n/q_n is the nth convergent to the continued fraction $\alpha=[a_0,a_1,\cdots]$ (α irrational, $\alpha>1$), then if $q_n\leqq y<q_{n+1}$, $|x-\alpha y|>1/(3a_{n+1}y)$. (ii) If $\alpha>1$ and $\beta>1$ are irrationals whose continued fraction expansions have their partial quotients less than N, then $|(x-\alpha y)(x+\beta y)|>1/(3N)$ for x, y not both zero.

G. Pall (Chicago, Ill.).

Prasad, A. V. A nonhomogeneous inequality for integers in a special cubic field. I, II. Nederl. Akad. Wetensch., Proc. 52, 240–250, 338–350 = Indagationes Math. 11, 55–65, 112–124 (1949).

Let $\theta, \phi, \bar{\phi}$ be the real and complex roots of the equation $t^3-t-1=0$; then $k(\theta)$ is the cubic field of smallest discriminant. Let a be any real number and let $b, \bar{b}$ be any conjugate complex numbers. The following theorem is proved. There is an integer ξ of $k(\theta)$ with conjugates $\eta, \bar{\eta}$ in $k(\phi)$, $k(\bar{\phi})$ for which $|(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})|<1/5.0001$, unless $a, b, \bar{b}$ are of the form

$$a=\frac{\sigma}{\theta^2+1}+\xi_1, \quad b=\frac{\tau}{\phi^2+1}+\eta_1, \quad \bar{b}=\frac{\bar{\tau}}{\bar{\phi}^2+1}+\bar{\eta}_1,$$

where $\xi_1, \eta_1, \bar{\eta}_1$ are conjugate integers and $\sigma, \tau, \bar{\tau}$ are conjugate units of $k(\theta)$, $k(\phi)$, $k(\bar{\phi})$. If $a, b, \bar{b}$ are of this form then there is no integer ξ in $k(\theta)$ with conjugates $\eta, \bar{\eta}$ for which the above absolute value is smaller than $\frac{1}{5}$, but there are an infinite number for which it is equal to $\frac{1}{5}$. To each triple of conjugates $\xi, \eta, \bar{\eta}$ there correspond rational integers u, v, w such that

$$\xi=u+v\theta+w\theta^2, \quad \eta=u+v\phi+w\phi^2, \quad \bar{\eta}=u+v\bar{\phi}+w\bar{\phi}^2,$$

and conversely, so that this theorem can also be regarded as a statement about the product of three special complex nonhomogeneous linear forms. From this point of view one is more interested in $M(\xi, \eta, \bar{\eta})$, the lower bound of numbers λ such that there are rational integers u, v, w for which $|(\xi-a)(\eta-b)(\bar{\eta}-\bar{b})|<\lambda|\Delta|$, where Δ is the determinant of the forms. The above theorem shows that for these special forms $M(\xi, \eta, \bar{\eta})=1/5\sqrt{23}$. This paper is closely related to that by Davenport [Proc. Cambridge Philos. Soc. 43, 137–152 (1947); these Rev. 8, 444] on products of real forms in cubic fields. *W. J. LeVeque* (Ann Arbor, Mich.).

Hlawka, Edmund. Ausfüllung und Überdeckung konvexer Körper durch konvexe Körper. Monatsh. Math. 53, 81–131 (1949).

Let $K=K(r, f)$ be a convex body in n-dimensional space, with the origin 0 as centre, defined by an inequality $f(x)=f(x_1, \cdots, x_n)\leqq r$. Let

$$M=\sup_{\Gamma} \inf_{0\neq x\in\Gamma} f(x), \quad \epsilon=\inf_{\Gamma} \sup_{z} \inf_{x\in\Gamma} f(x+z),$$

where Γ is an arbitrary lattice with determinant 1 and x and z are arbitrary points. The author proves that

$$1\leqq[1-\tfrac{1}{2}(1-\tfrac{1}{2}J^{1/n}M)^n]^{-1}\leqq\epsilon^nJ\leqq(\tfrac{1}{2}n)^nM^nJ\leqq n^n,$$

where J is the volume of $K(1, f)$.

A set of bodies $K_i: f(x-p_i)\leqq r$, $i=1, \cdots, a$, is said to form a packing of a bodies K into a convex body B, if each K_i is contained in B and no two K_i have common inner points. A similar set K_i, $i=1, \cdots, u$, is said to form a covering of B by u bodies K if each K_i contains an inner point of B and every point of B is contained in some K_i. A packing of K into B is said to be a lattice packing, if, for some lattice Γ and some point z, (i) $p_i=g_i+z$ for some g_i of Γ, and (ii) if $p+z\epsilon\Gamma$ and the body $f(x-p)\leqq r$ is contained in B, then p is one of the p_i. [The author omits the essential condition (ii); his definitions 5, 13 and 24 need appropriate modifications, and so do the proofs of theorems 5 and 26.] A lattice covering is defined similarly. Let $a(B, K)$ and $u(B, K)$ be the upper and lower bounds of the number of bodies K in the packings and coverings of B, respectively. Let $a^*(B, K)$ and $u^*(B, K)$ be the corresponding bounds for the lattice packings and lattice coverings.

The author obtains both upper and lower bounds for both $a^*(B, K)$ and $u^*(B, K)$, which ensure that, if V is the volume of B, the limits

$$\Delta^*=\lim_{r\to 0} a^*(B, K)Jr^n/V, \quad \vartheta^*=\lim_{r\to 0} u^*(B, K)Jr^n/V$$

exist and have the values $(\frac{1}{2}M)^nJ$ and ϵ^nJ. Further it is shown that

$$\Delta=\lim_{r\to 0} a(B, K)Jr^n/V, \quad \vartheta=\lim_{r\to 0} u(B, K)Jr^n/V$$

exist and are independent of B.

It is clear that $\Delta^*\leqq\Delta\leqq 1\leqq\vartheta\leqq\vartheta^*$. Improved upper bounds for Δ and lower bounds for ϑ are found for certain classes of bodies K similar to those discussed by J. G. van der Corput and H. Davenport [Nederl. Akad. Wetensch., Proc. 49, 701–707 = Indagationes Math. 8, 409–415 (1946); these Rev. 8, 317]. Other topics discussed are: upper bounds for Δ for spheres and cylinders; packings and coverings with convex bodies which do not have a centre; distributions of points $p_1, \cdots, p_s$ in B with $f(p_i-p_j)\geqq d$, if $i\neq j$; packings in which at most $k-1$ bodies may overlap; coverings in which each point of B belongs to at least k bodies; packings and coverings with bodies, which are not all the same size, or which are selected from a set of bodies, not all the same shape; selection from coverings of systems of nonoverlapping bodies. *C. A. Rogers* (Princeton, N. J.).

Hlawka, Edmund. Ausfüllung und Überdeckung durch Zylinder. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. 85, 116–119 (1948).

Let $f(x_1, \cdots, x_n)=\max\{f_1(x_1, \cdots, x_m), f_2(x_{m+1}, \cdots, x_n)\}$, where f, f_1 and f_2 are the distance functions of symmetrical convex bodies K, K_1 and K_2 in n, m and $n-m$ dimensional spaces. Let Δ and ϑ, Δ_1 and ϑ_1, and Δ_2 and ϑ_2 be the densities defined, as in the preceding review, for the bodies K, K_1 and K_2. The author proves that

$$\Delta_1\Delta_2\leqq\Delta\leqq\min(\Delta_1, \Delta_2), \quad \vartheta_1\vartheta_2\geqq\vartheta\geqq\max(\vartheta_1, \vartheta_2),$$

and discusses some simple consequences of these results. *C. A. Rogers* (Princeton, N. J.).

Whitworth, J. V. On the densest packing of sections of a cube. Ann. Mat. Pura Appl. (4) 27, 29–37 (1948).

The author considers the densest lattice packing of convex bodies K of the form

$$\text{(A)} \qquad |x|\leqq 1, \quad |y|\leqq 1, \quad |z|\leqq 1, \quad |x+y+z|\leqq\tau,$$

where $0<\tau<3$, and gives, without proofs, a summary of the results which he has obtained. His method is based on Minkowski's work and, for $\tau=1$, includes Minkowski's result concerning the densest packing of octahedra [Gesammelte Abhandlungen, v. 2, pp. 1–42 = Nachr. Ges. Wiss. Göttingen 1904, 311–355]. For such a densest packing the lattice formed by the centres of the bodies K must be K-admissible, i.e., no point of the lattice different from the origin is an inner point of (A), and must be critical, i.e., its determinant $d(\Lambda)$ equals the lower bound $\Delta(K)=\inf d(\Lambda)$ extended over all K-admissible lattices Λ. The packing constant $q(K)=\frac{1}{4}V(K)/\Delta(K)$ where $V(K)$ is the volume of K.

The investigation falls into three parts according as (a) $0<\tau\leqq\frac{1}{2}$, (b) $\frac{1}{2}\leqq\tau\leqq 1$, (c) $1\leqq\tau<3$ and the critical lattices are determined in each case. The main results are as follows:

(a) $\Delta(K)=\frac{1}{4}\tau$, $\quad q(K)=\frac{1}{6}(9-\tau^2)$,

(b) $\Delta(K)=-\frac{1}{32}\tau(\tau^3+3\tau^2-24\tau+1)$,

$\qquad q(K)=\frac{1}{4}\tau(9-\tau^2)/(-\tau^3-3\tau^2+24\tau-1)$,

(c) $\Delta(K)=\frac{1}{27}\tau(\tau^3-9\tau+27)$,
$q(K)=\frac{1}{8}(\tau^3-9\tau^2+27\tau-3)/\tau(\tau^3-9\tau+27)$.

In cases (b) and (c) there is only one critical lattice for each value of τ, but there are an infinity of critical lattices in case (a) for each τ. The packing constant $q(K)$ is a minimum for $\tau=1$, so that, of the bodies considered, the octahedra can be packed least densely. *R. A. Rankin.*

Mahler, Kurt. On the successive minima of a bounded star domain. Ann. Mat. Pura Appl. (4) **27**, 153–163 (1948).

Let $F(X)$ be a continuous distance function, defined for the points of a plane, which defines a bounded star domain K of points X for which $F(X)\leqq 1$. Let P, Q run through all point pairs of the lattice Λ which generate a lattice and for which $F(P)\leqq F(Q)$. Let $\mu_1(\Lambda)$, $\mu_2(\Lambda)$ be defined to be the minima of $F(P)$ and $F(Q)$, respectively, while $\mu(\Lambda)$ is defined to be the product $\mu_1(\Lambda)\mu_2(\Lambda)$. Let $M(K)$ be the upper bound of $\mu(\Lambda)$, where Λ runs through all lattices whose determinants are equal to $\Delta(K)$. A lattice Λ for which $\mu(\Lambda)=M(K)$ is defined to be extreme. The principal result of the paper is that an extreme lattice exists for every plane bounded symmetric star body K. The author also discusses some of the relationships between extreme and critical lattices. For the latter it follows easily that $\mu(\Lambda)=1$ and so if $M(K)=1$ every critical lattice is extreme. This is always the case if K is convex. A star body K is constructed for which $M(K)>1$ which means that, in this case, the critical lattices are not extreme. *D. Derry.*

Nosarzewska, M. Évaluation de la différence entre l'aire d'une région plane convexe et le nombre des points aux coordonnées entières couverts par elle. Colloquium Math. **1**, 305–311 (1948).

Let I be a plane open convex region of area a which is bounded by a closed curve J of length l, and let w be the number of points with integral coordinates covered by I. The author proves that $-\frac{1}{2}l-1<a-w<\frac{1}{2}l$. This is done by dividing the plane into a mesh of squares centred at the lattice points. Difficulties arise only for squares which contain points of J, and such squares are divided into several different types which are considered separately. The isoperimetric property of the circle and the convexity of J are used to obtain the desired inequalities. The author shows that her inequalities cannot be improved. This follows, for the left-hand inequality, by considering the sequence of squares I_n with a pair of opposite vertices at $(-1/n, -1/n)$, $(m+1/n, m+1/n)$. Here m is a fixed non-negative integer and $n=1, 2, \cdots$. The squares I_n may be taken arbitrarily small or arbitrarily large by suitable choice of m. For the right-hand inequality she considers a diminishing sequence of squares of side $1/n$ which do not contain any lattice point ($n=1, 2, \cdots$). It is not known if the right-hand inequality remains the best possible for arbitrarily large regions I. *R. A. Rankin* (Cambridge, England).

Shapiro, George. On the non-vanishing at $s=1$ of certain Dirichlet series. Amer. J. Math. **71**, 621–626 (1949).

Suppose (i) $f(n)$ is a completely multiplicative function, (ii) $|f(n)|=0$ or 1 for each n, so that

$$L(s)=\sum_1^\infty f(n)n^{-s}=\prod_p(1-f(p)p^{-s})^{-1} \quad (\sigma=\Re s>1),$$

(iii) $L(s)$ (regular in $\sigma>1$) admits a regular analytical continuation along the real segment $\frac{1}{2}\leqq s\leqq 1$. Then $L(1)\neq 0$. [For this result, which includes all cases of the fundamental theorem on the nonvanishing of Dirichlet's functions $L(s, \chi)$ on the line $\sigma=1$, see Ingham, J. London Math. Soc. **5**, 107–112 (1930).] For real $f(n)$ Wintner generalised (ii) to (ii'): $f(p)\geqq -1$ for each prime p, and $\sum f^2(n)n^{-s}$ is convergent for $\sigma>1$ [same J. **68**, 285–292 (1946); these Rev. **7**, 416]. The aim of the present paper is to relax (ii) without restricting $f(n)$ to be real. Thus theorem I asserts that (ii) may be replaced by (ii''): $f(n)$ is bounded (which means, with (i), that $|f(n)|\leqq 1$). The attempt to extend the discussion to unbounded (complex) $f(n)$ does not lead to a direct generalisation of Wintner's theorem, but yields a somewhat different type of result (theorem II) in which (iii) is relaxed at the cost of more elaborate forms of restriction in (ii). *A. E. Ingham* (Cambridge, England).

Majumdar, Kulendra N. On the parity of the partition function $p(n)$. J. Indian Math. Soc. (N.S.) **13**, 23–24 (1949).

The function $p(n)$, for n a positive integer, denotes the total number of unrestricted partitions of n; $p(0)=1$; and for all other values of n, $p(n)$ is taken as zero. No really satisfactory answer has ever been given to the question of whether or not $p(n)$ is odd, given a value of n. MacMahon has suggested the use of the congruence

$$p(n)\equiv\sum_{m=0}^{\infty}p\{(2n-m^2-m)/8\} \pmod 2$$

by means of which the parity of $p(n)$ may be determined when those of $p(m)$ are known for $m\leqq n/4$. The author suggests as an alternative the congruence

$$p(n)\equiv\sum_t p((8n-t+5)/128) \pmod 2$$

where t ranges over integers of the form $sq^{4\alpha+1}$ where α is a nonnegative integer and s is an odd square, prime to the prime $q=8k+5$. Although this formula involves a knowledge of the parities of $p(m)$ for $m\leqq n/16$ only, it has many more terms than the MacMahon congruence, and is more difficult to apply. *D. H. Lehmer* (Berkeley, Calif.).

Redei, L., and Ren'i, A. On the representation of the numbers $1, 2, \cdots, N$ by means of differences. Mat. Sbornik N.S. **24**(66), 385–389 (1949). (Russian)

[The authors' names appear in non-Russian publications as Rédei and Rényi.] Let $H(n)$ denote the set $\{1, 2, \cdots, n\}$. A set B of integers is called a basis of $H(n)$ if, given any $k\in H(n)$, the equation $k=a_i-a_j$ is soluble for some $a_i, a_j\in B$. The smallest possible number of elements in a basis of $H(n)$ is denoted by n^*, and the problem is to investigate the behaviour of n^* as $n\to\infty$. It is almost trivial that $\sqrt{2}\leqq n^*/\sqrt{n}\leqq 2+1/\sqrt{n}$, for $n\leqq\binom{n^*}{2}$ and $\{1, 2, 3, \cdots, k, 2k, 3k, \cdots, k^2\}$ is a basis of $H(k^2-1)$. The authors show that, in fact, $\lim_{n\to\infty} n^*/\sqrt{n}$ exists, and

$$\left(2+\frac{\sqrt{4}}{3\pi}\right)^{\frac{1}{2}}\leqq\lim_{n\to\infty}\frac{n^*}{\sqrt{n}}\leqq\left(\frac{8}{3}\right)^{\frac{1}{2}}. \tag{1}$$

Proof. Take any integer $\nu\geqq 1$, any $\epsilon>0$, and some $\vartheta>0$ such that $\vartheta\nu^*/\sqrt{\nu}<\epsilon/2$. Then a number $n_0(\nu, \epsilon)$ can be found such that, for $n>n_0(\nu, \epsilon)$, $\nu^*/\sqrt{n}<\epsilon/2$ and there exists some prime q in the range $(n/\nu)^{\frac{1}{2}}<q<(1+\vartheta)(n/\nu)^{\frac{1}{2}}$. Write $m=q^2+q+1$. By a theorem due to J. Singer [Trans. Amer. Math. Soc. **43**, 377–385 (1938); see also T. Vijayaraghavan and S. Chowla, Proc. Nat. Acad. Sci. India. Sect. A. **15**, 194 (1945); these Rev. **7**, 505] there exist numbers

$a_0, a_1, \cdots, a_q$ such that, given any a in $0 \leqq a < m$, either $a = a_i - a_j$ or $a - m = a_i - a_j$ is soluble. If now $\{b_1, \cdots, b_{\nu^*}\}$ denotes a basis of $H(\nu)$, then the set $\{a_i + mb_k\}$ $(0 \leqq i \leqq q;\ 1 \leqq k \leqq \nu^*)$ is a basis of $H(m\nu)$, and so $(m\nu)^* \leqq (q+1)\nu^*$. But $n < m\nu$, and therefore

$$n^* \leqq (q+1)\nu^* < \left\{(1+\vartheta)\left(\frac{n}{\nu}\right)^{\frac{1}{2}} + 1\right\}\nu^*.$$

Thus, for any ν, any $\epsilon > 0$, and $n > n_0(\nu, \epsilon)$ we have $n^*/\sqrt{n} \leqq \nu^*/\sqrt{\nu} + \epsilon$; hence $n^*/\sqrt{n}$ converges to its precise lower bound, and the right-hand inequality in (1) now follows in view of $6^* = 4$.

To complete the proof write $k = n^*$ and let $\{b_1, \cdots, b_k\}$ be a basis of $H(n)$. Then, for $1 \leqq |t| \leqq n$, the equation $t = b_r - b_s$ $(1 \leqq r, s \leqq k)$ has at least one solution. Hence

$$2\sum_{t=1}^{n} \cos tx + k^2 - 2n \geqq \sum_{r,s=1}^{n} \cos (b_r - b_s)x = \left|\sum_{r=1}^{k} e^{ib_r x}\right|^2 \geqq 0.$$

But, for $x = 3\pi/(2n+1)$,

$$2\sum_{t=1}^{n} \cos tx < -\frac{2}{3\pi}(2n+1) - 1.$$

Hence $(2n+1)(1+2/3\pi) < n^{*2}$, and the required result follows at once. [The reviewer observes that the first inequality on line 5 of p. 388 should be $\vartheta\nu^*/\sqrt{\nu} < \epsilon/2$.]

L. Mirsky (Sheffield).

Erdös, P., and Gál, I. S. On the representation of $1, 2, \cdots, N$ by differences. Nederl. Akad. Wetensch., Proc. **51**, 1155–1158 = Indagationes Math. **10**, 379–382 (1948).

A set of integers $a_1, a_2, \cdots, a_{k(n)}$ is called a difference-basis with respect to n if every integer m with $0 < m \leqq n$ can be represented in the form $a_i - a_j$. The set is called a restricted difference-basis if all the elements a_i satisfy the condition $0 \leqq a_i \leqq n$. Denote by $s^* = s^*(n)$ the minimum of $k(n)$ for a basis with respect to n and by $s = s(n)$ the minimum of $k(n)$ for a restricted basis with respect to n. It was proved by the reviewer [J. Elisha Mitchell Sci. Soc. **61**, 55–66 (1945); these Rev. **7**, 47] that for every given integer s there exists a restricted difference basis of s elements with respect to n such that $n \geqq k^2/4 + 7k/6 - 53/12$. Rédei and Rényi [see the preceding review] proved that $\lim_{n\to\infty} s^*/n^{\frac{1}{2}}$ exists and equals the greatest lower bound of $s^*/n^{\frac{1}{2}}$. Moreover,

$$\{2+4/(3\pi)\}^{\frac{1}{2}} \leqq \lim_{n\to\infty} s^*/n^{\frac{1}{2}} \leqq \left(\frac{8}{3}\right)^{\frac{1}{2}}.$$

The authors prove the corresponding results for $s(n)$, using the paper of Rédei and Rényi which was known to them before it was published. *A. Brauer* (Chapel Hill, N. C.).

Erdös, P., and Koksma, J. F. On the uniform distribution modulo 1 of lacunary sequences. Nederl. Akad. Wetensch., Proc. **52**, 264–273 = Indagationes Math. **11**, 79–88 (1949).

Let $0 \leqq \alpha < \beta \leqq 1$. Denote by $u_1, u_2, u_3, \cdots$ a sequence of real numbers, by N' the number of those elements of the set

$$u = u_1 - [u_1], u_2 - [u_2], \cdots, u_N - [u_N]$$

satisfying $\alpha \leqq u < \beta$. The upper bound of $|N'/N - (\beta - \alpha)|$ for variable α, β and fixed $N \geqq 1$ is the discrepancy $D(N)$ of the sequence; it satisfies $D(N) = o(1)$ if and only if the sequence is uniformly distributed [Koksma, Diophantische Approximationen, Ergebnisse der Math., v. 4, no. 4, Springer, Berlin, 1936, chaps. 8, 9]. The authors prove a very general result for sequences $u_n = f(n, \theta)$ and nearly all θ; it contains the following special theorem. Let $a < b$, $\delta > 0$. Let $u_n = f(n, \theta)$ denote a sequence of real functions defined for $a \leqq \theta \leqq b$ satisfying

$$f_\theta'(n+1, \theta) \geqq (1+\delta) f_\theta'(n, \theta) > 0;$$
$$f_\theta''(n+1, \theta) \geqq (1+\delta) f_\theta''(n, \theta) \geqq 0$$

$(n = 1, 2, \cdots)$ for all θ in $a \leqq \theta \leqq b$. Let $\omega(n)$ be an increasing function of n tending to infinity with n. Then

$$D(n) = o\{N^{-\frac{1}{2}}(\log N)^{\frac{1}{2}}(\log \log n)^{\frac{1}{2}}\omega(N)\}$$

for nearly all θ in $a \leqq \theta \leqq b$. *K. Mahler* (Manchester).

Ostmann, Hans-Heinrich. Über die Dichten additiv komponierter Zahlenmengen. Arch. Math. **1**, 393–401 (1949).

This is an expository paper giving the history of the problems and results obtained thus far on the density and asymptotic density of sums of sets of positive integers. The author also discusses results previously obtained by himself on some variations of these concepts [Deutsche Math. **6**, 213–247 (1941); these Rev. **8**, 194]. He also announces an improvement over P. Erdös's result on the asymptotic density of the sum of two sets of positive integers. Detailed proofs of the author's new result are to appear later.

H. B. Mann (Berkeley, Calif.).

Chowla, S. D., and Todd, John. The density of reducible integers. Canadian J. Math. **1**, 297–299 (1949).

Let $P(m)$ denote the greatest prime factor of m. The authors call an integer n reducible if $P(n^2+1) < 2n$, and they adduce numerical evidence in support of the conjecture that the set of reducible integers possesses a density which is approximately equal to 0.3. They also prove that the set of integers n such that $P(n) < 2n^{\frac{1}{2}}$ has density $1 - \log 2$. This result is obtained by estimating in turn the number of positive integers $n \leqq x$ for which $P(n) > 2x^{\frac{1}{2}}$ and $2n^{\frac{1}{2}} \leqq P(n) \leqq 2x^{\frac{1}{2}}$ respectively, and using in both cases the classical asymptotic formula for $\sum_{p \leqq x} p^{-1}$. The authors observe, moreover, that a similar argument will show that the density of the set of numbers n such that $P(n) > An^{\alpha}$, where $A \geqq 1$, $\frac{1}{2} \leqq \alpha < 1$, is $\log(1/\alpha)$. [This is erroneously stated as $\log \alpha$ in the paper. The authors have informed the reviewer that the table on page 297 contains numerous minor errors, and they have supplied an amended table. They also point out that the words "right" and "left" on page 297, § 1, lines 4 and 5, should be interchanged.]

L. Mirsky (Sheffield).

Mirsky, L. A property of square-free integers. J. Indian Math. Soc. (N.S.) **13**, 1–3 (1949).

Let $q_1, q_2, \cdots, q_s$ be any nonzero integers. Then the set of those positive integers n for which all the numbers $n^2+q_1, n^2+q_2, \cdots, n^2+q_s$ are square-free is either empty or has positive lower density. *T. Estermann* (London).

Mirsky, L. Generalizations of a problem of Pillai. Proc. Roy. Soc. Edinburgh. Sect. A. **62**, 460–469 (1949).

Pillai [J. Indian Math. Soc. (N.S.) **2**, 116–118 (1936)] gave an asymptotic formula for the number of positive integers $n \leqq x$ such that $n+k_1, \cdots, n+k_s$ are square-free, $k_1, \cdots, k_s$ being fixed distinct positive integers. The corresponding problem for r-free integers was considered by Mirsky [Quart. J. Math., Oxford Ser. **18**, 178–182 (1947); these Rev. **9**, 80]. In the present paper the problem of Pillai is generalized as follows. Let $\mathbf{A}$ be a given set of integers. Numbers of the form $a_1^{u_1} \cdots a_k^{u_k}$, where $a_1, \cdots, a_k \in \mathbf{A}$, will

be called c-numbers. Let $C_1, \cdots, C_s$ be given sets of c-numbers and denote by $\mathfrak{G}(n)$ the greatest c-number dividing n. Further, let **M** be the set of integers n for which $\mathfrak{G}(n+k_1)\varepsilon C_1, \cdots, \mathfrak{G}(n+k_s)\varepsilon C_s$, and $M(x)$ be the number of integers $n \leqq x$ belonging to **M**. An asymptotic formula, for $x \to \infty$, is obtained for $M(x)$ and the density of **M** is discussed under various assumptions regarding the integers of A. When each C_i consists of the single number 1, **M** consists of the integers n such that $n+k_1, \cdots, n+k_s$ are **A**-free (i.e., not divisible by any integer of **A**). *W. H. Simons.*

Mirsky, L. Summation formulae involving arithmetic functions. Duke Math. J. **16**, 261–272 (1949).

Let $F_1, \cdots, F_s$ be arithmetic functions, and $k_1, \cdots, k_s$ given fixed integers. The author obtains general asymptotic formulae, for $x \to \infty$, for sums of the form

$$\sum_{0<n\leq x} F_1(n+k_1) \cdots F_s(n+k_s)$$

under various sets of assumptions about $F_1, \cdots, F_s$. The general results are used to obtain asymptotic expressions for sums of the above type when the F's are (a) functions $\mu_k(n)$ defined to be 1 or 0 according as n is or is not k-free, (b) Euler's φ-function $\varphi(n)$. *W. H. Simons.*

Atkinson, F. V., and Cherwell. The mean-values of arithmetical functions. Quart. J. Math., Oxford Ser. **20**, 65–79 (1949).

Let $\{A_n\}$, $\{a_n\}$ be two sequences connected by the relation $A_n = \sum_{m|n} a_m$. The authors first prove that if $\lambda(n)$ is a positive nondecreasing function of n, such that $\lambda(n)/\log n$ is nonincreasing, and if $\sum_1^N a_n = o(N/\lambda(n))$, $\log(N^{-1}\sum_1^N |a_n|) = O(\lambda(N))$, then $N^{-1}\sum_1^N A_n \to \sum_1^N a_n/n$ as $N \to \infty$. Next, this result is extended to cover the case when n runs through an arithmetic progression. Finally, a generalized theorem about the average of products of the A_n when n runs through a polynomial sequence is established. The results are derived by elementary methods and include many of the standard mean-value and distribution formulae of the theory of numbers. Among the numerous applications the following theorem may be cited. The number of kth-power-free numbers not exceeding N, of the form n^l+h, where $h \neq 0$, is asymptotic to $N^{1/l}\prod((1-(\chi(p)/p^k))$, provided that $k \geqq l$, $k \geqq 2$. Here $\chi(m)$ denotes the number of solutions of the congruence $x^l+h \equiv 0 \pmod{m^k}$. A corresponding estimate for numbers of the form p^l+h, where p is a prime, is based upon results related to the prime number theorem. *A. L. Whiteman* (Los Angeles, Calif.).

Davenport, H. A divisor problem. Quart. J. Math., Oxford Ser. **20**, 37–44 (1949).

Let $\sigma(n)$ denote the sum of the divisors of n, and consider the problem of determining a bound for the error term $R(x)$ in the formula $\sum_{n\leq x}\sigma(n) = \pi^2x^2/12 + xR(x)$. If $\rho(x) = \sum_{m\leq x}\psi(x/m)/m$, $\psi(u) = u - [u] - \frac{1}{2}$, it is easy to prove that $R(x) = -\rho(x) + O(1)$. The inequality $\rho(x) = O(\log x)$ is obvious. Using Weyl's inequality for exponential sums, Walfisz improved this to $\rho(x) = O(\log x/\log\log x)$, and then to $O(\log x)^{1+\varepsilon}$ by applying the more refined inequalities of Vinogradov [references are given in the paper]. Davenport gives another proof of this result, also using Vinogradov's inequalities. *R. Bellman* (Stanford University, Calif.).

Jones, B. W. A theorem on integral symmetric matrices. Bull. Amer. Math. Soc. **55**, 620–622 (1949).

A proof is given of the following theorem. If A and B are integral, nonsingular symmetric matrices, of orders n and m ($n > m$), and if S is a rational representation of B by A, then there exists a rational n by n matrix T, whose denominators contain at most prime factors which appear in the denominators of S, such that $|T| = 1$ and T replaces A by a matrix A_0 which represents B integrally. The essential idea of the proof is a canonicization of S, rather than A or B. From Jones's theorem follows in particular a theorem of Siegel [Ann. of Math. (2) **36**, 527–606 (1935), lemma 24] that if a number or form is represented rationally, but with denominators prime to $2|A|$, by an integral matrix A, then that number or form is represented integrally by some form in the genus of A. Also, if a classic quadratic form f represents an integer or classic form g rationally, then some other classic form f' of the same determinant as f represents g integrally. *G. Pall* (Chicago, Ill.).

Walton, Jean B. Theta series in the Gaussian field. Duke Math. J. **16**, 479–491 (1949).

Let $\vartheta_r(\tau, \rho, 1) = \sum \mu^r \exp(2\pi i \tau \mu\mu'/4)$, where the sum is taken over all Gaussian integers $\mu \equiv \rho \pmod 4$ and r is a positive integer. Only even values of r are of interest. For $r \equiv 0 \pmod 4$ there are three independent functions given by taking $\rho = 0$, $1+i$ and 1. For $r \equiv 2 \pmod 4$ all the functions can be expressed in terms of $\vartheta_r(\tau, 1, 1)$. The behavior of each of these functions under modular transformations on τ is determined. In general the functions are not modular but $\omega_2^{-(r+1)}\vartheta_r(\omega_1/\omega_2, \rho, 1)$ is a modular form of dimension $-(r+1)$ and level 4. Now let $f_r(\tau) = \sum \mu^r \exp(\pi i \tau \mu\mu')$ where μ runs through all Gaussian integers. The only case of interest is $r \equiv 0 \pmod 4$. Then $f_r(\tau)$ can be expressed in terms of the $\vartheta_r(\tau, \rho, 1)$. This makes it possible to obtain transformation formulas for $f_r(\tau)$ and then to express $f_r(\tau)$ in terms of Jacobi's theta functions. This is done and the explicit expressions are given for certain values of r. Finally there is a discussion of the multiplicative properties of the coefficients of $f_r(\tau)$ when it is expanded in powers of $e^{\pi i \tau}$. *H. S. Zuckerman* (Seattle, Wash.).

Litver, E. L. On the number of ideal classes of certain special fields. Doklady Akad. Nauk SSSR (N.S.) **66**, 335–338 (1949). (Russian)

Soient p un nombre premier, k un corps de nombres algébriques dont tous les idéaux sont principaux, $\alpha_1, \cdots, \alpha_s$ un ensemble fini de nombres non-nuls de k multiplicativement indépendants par rapport au groupe des puissances p-ièmes des $\alpha \varepsilon k$ non-nuls, $K = k(\alpha_1^{1/p}, \alpha_2^{1/p}, \cdots, \alpha_s^{1/p})$. L'auteur démontre que le nombre des classes d'idéaux de K ne diffère que par un facteur puissance de p du produit des nombres des classes d'idéaux de tous les sous-corps distincts de K, qui sont de la forme $k(\beta)^{1/p}$. Ce résultat, qui généralise celui de Dirichlet pour les corps biquadratiques, n'était connu que pour $p = 2$. Idée de démonstration: soient $K_1, \cdots, K_s$ tous les corps de la forme $k(\beta^{1/p})$, contenus dans K. Soient H_i le groupe des classes d'idéaux de K_i et G_i le plus grand sous-groupe de H_i d'ordre premier à p. Si h_i, g_i sont les ordres des H_i, G_i, $h_1h_2 \cdots h_s/g_1g_2 \cdots g_s$ est une puissance de p. Si $A_i \varepsilon G_i$, soit $\bar{A}_i$ la classe d'idéaux de K contenant A_i. Si $\bar{A}_1 \cdots \bar{A}_s = I$, il existe des idéaux $\mathfrak{A}_i \varepsilon A_i$ des K_i et un nombre α de K tels que $\mathfrak{A}_1\mathfrak{A}_2 \cdots \mathfrak{A}_s = (\alpha)$. Si $N_i(\cdots)$ désigne la norme de K à K_i, on voit facilement que $N_i(\mathfrak{A}_i) = \mathfrak{A}_i^{p^{n-1}}$, tandis que, si $j \neq i$, on a

$$N_i(\mathfrak{A}_j) = (N_{K_j/k}(\mathfrak{A}_j))^{p^{n-2}},$$

où $N_{K_j/k}(\mathfrak{A}_j)$ est un idéal de k, donc un idéal principal. De même, $N_i((\alpha)) = (N_i(\alpha))$ l'est. Donc on a, dans K_i, $\mathfrak{A}_i^{p^{n-1}} \sim I$, d'où $A_i^{p^{n-1}} = I$ et $A_i = I$. Par suite, $\bar{A}_1\bar{A}_2 \cdots \bar{A}_s = I$

entraîne, pour tout i, $A_i=I$, et le groupe G des classes d'idéaux de K, engendré par tous les $\bar{A}_i$ ($A_i \varepsilon G_i, i=1, 2, \cdots, s$), a l'ordre $g_1 g_2 \cdots g_s$. On montre par induction que, pour tout idéal $\mathfrak{A}$ de K, $\mathfrak{A}^{p^{n-1}}$ équivaut à un produit d'idéaux des K_i. En effet, supposant l'affirmation prouvée pour tous les corps entre k et K, considérons un sous-corps K'' de K tel que $(K:K'')=p^2$. Alors, il existe deux nombres μ_1, μ_2 de k tels que $K=K''(\mu_1^{1/p}, \mu_2^{1/p})$, et il y a $p+1$ corps K' entre K'' et K, tels que $(K:K')=p$, à savoir $K_0'=K''(\mu_1^{1/p})$ et les $K_r'=K''((\mu_1 r \mu_2)^{1/p})$ $(r=1, \cdots, p)$. Pour $r=0, 1, \cdots, p$, $N_{K/K_r'}(\mathfrak{A})$ est un idéal de K_r', et il est facile de voir que le produit de tous les $N_{K/K_r'}(\mathfrak{A})$ $(r=0, 1, \cdots, p)$ est le produit de $\mathfrak{A}^p$ par $N_{K/K''}(\mathfrak{A})$, qui est un idéal de K''. Ainsi, $\mathfrak{A}^p$ équivaut à un produit des idéaux des corps K_0', K_1', $\cdots$, K_p', K'', qui sont tous entre k et K. Ces idéaux sont donc tels que leur p^{n-2}-ième puissance équivaut au produit des idéaux de certains des K_i, ce qui achève la démonstration. Mais, alors, $\mathfrak{A}^{p^{n-1}h_1h_2\cdots h_s/g_1g_2\cdots g_s}$ est dans une $H \varepsilon G$. Par suite, H étant le groupe des classes d'idéaux de K, et h étant son ordre, $h/g_1g_2\cdots g_s=(H:G)$ est une puissance de p, et il en est de même pour $h/h_1h_2\cdots h_s=(H:G)$ $(h_1h_2\cdots h_s/g_1g_2\cdots g_s)$.

M. Krasner (Paris).

ANALYSIS

Taussky, Olga. A remark concerning the characteristic roots of the finite segments of the Hilbert matrix. Quart. J. Math., Oxford Ser. **20**, 80–83 (1949).

The matrix in question is the infinite matrix (a_{ij}) with $a_{ij}=(i+j)^{-1}$. It was known that the greatest characteristic root λ_n of the submatrix formed from the first n rows and columns tends to π as $n\to\infty$; this is a sharpened form of Hilbert's inequality. [See Frazer, J. London Math. Soc. **21**, 7–9 (1946); Cassels, ibid. **23**, 285–290 (1948); these Rev. **8**, 259; **10**, 434; and for numerical data, Copsey, Frazer and Sawyer, Nature **161**, 361 (1948); Math. Gaz. **32**, iii–iv (1948); Fairthorne and Miller, Math. Tables and Other Aids to Computation **3**, 399–400 (1949); these Rev. **9**, 344; **10**, 367, 626.] The author shows that $\lambda_n=\pi+O(1/\log n)$, by considering $\sum_{i,j=1}^{n}(i+j)^{-1}x_ix_j$ with $x_k=k^{-\frac{1}{2}}$.

R. P. Boas, Jr. (Providence, R. I.).

Vitvickiĭ, N. K. A generalization of Cauchy's theorem in the calculus of finite differences. Tomsk. Gos. Univ. Učenye Zapiski **1948**, no. 8, 3–7 (1948). (Russian)

The following lemma of S. Bernstein [Math. Ann. **75**, 449–468 (1914)] is a generalization of Rolle's theorem to finite differences. If $F(x)$ is continuous in the interval $a\leqq x\leqq b$, and vanishes at $n+1$ points of the interval, then for any sufficiently small positive value h the nth order finite difference $\Delta_h^nF(x)$, defined by

$$\Delta_h^1F(x)=F(x+h)-F(x), \quad \Delta_h^nF(x)=\Delta_h^1\Delta_h^{n-1}F(x), \qquad n>1,$$

has at least one zero in the interval; that is, we have $\Delta_h^nF(\xi)=0$ for some ξ with $a<\xi<b$. By means of Bernstein's lemma, D. Raikov [Rec. Math. [Mat. Sbornik] N.S. **7(49)**, 379–383 (1940); these Rev. **2**, 254] established the following generalization of Lagrange's law of the mean to divided differences. If $F(x)$ is continuous in the interval $a\leqq x\leqq b$, then for any distinct points $x_0, x_1, \cdots, x_n$ in the interval, and for any sufficiently small positive value h, the nth order divided difference $[x_0, x_1, \cdots, x_n; F]$, defined by

$$[x_0; F]=F(x_0), \quad [x_0, x_1, \cdots, x_n; F]=\frac{[x_1, x_2, \cdots, x_n; F]-[x_0, x_1, \cdots, x_{n-1}; F]}{x_n-x_0}, \qquad n>0,$$

satisfies $[x_0, x_1, \cdots, x_n; F]=\Delta_h^nF(\xi)/(n!h^n)$ for some ξ with $\min x_j<\xi<\max x_j$, $j=0, 1, \cdots, n$.

The author now uses Bernstein's lemma, and a known relation between divided differences and Vandermonde determinants, to establish the following analogous generalization of Cauchy's generalized law of the mean. If $F(x)$ and $G(x)$ are continuous in the interval $a\leqq x\leqq b$, the divided difference $[x_0, x_1, \cdots, x_n; G]$ does not vanish, and the finite differences $\Delta_h^nF(x)$ and $\Delta_h^nG(x)$ do not vanish simultaneously, then for any distinct points $x_0, x_1, \cdots, x_n$ in the interval, and for any sufficiently small positive h, we have

$$[x_0, x_1, \cdots, x_n; F]/[x_0, x_1, \cdots, x_n; G]=\Delta_h^nF(\xi)/\Delta_h^nG(\xi)$$

for some ξ with $\min x_j<\xi<\max x_j$, $j=0, 1, \cdots, n$.

E. F. Beckenbach (Los Angeles, Calif.).

Fenyö, Stefan. Über den Mischalgorithmus der Mittelwerte. Acta Univ. Szeged. Sect. Sci. Math. **13**, 36–42 (1949).

A continuous, strictly increasing, reflexive, and symmetric function $M(x_1, \cdots, x_n)$ is said to be a mean-value function; if further there is a continuous and strictly increasing function $\Phi(x)$ such that

$$M(x_1, \cdots, x_n)=\Phi^{-1}[\{\Phi(x_1)+\cdots+\Phi(x_n)\}/n],$$

then M is said to be quasi-arithmetic. For given mean-value functions $M^{(i)}$, $i=1, \cdots, n$, the algorithm $x_{i0}=x_i$; $x_{ik}=M^{(i)}(x_{1,k-1}, \cdots, x_{n,k-1})$; $i=1, \cdots, n$; $k=1, 2, \cdots$, can be shown to have the property that there is a function $X=M(x_1, \cdots, x_n)$ such that $\lim_{k\to\infty}x_{ik}=X$, independent of i. Clearly M is a mean-value function, which satisfies $M(x_1, \cdots, x_n)=M[M^{(1)}(x_1, \cdots, x_n), \cdots, M^{(n)}(x_1, \cdots, x_n)]$. The author seeks the subclass of mean-value functions $M^{(i)}$ for which the resultant M is quasi-arithmetic, and obtains the following necessary and sufficient condition:

$$\sum_{j=1}^{n}E(M^{(j)})\partial M^{(j)}/\partial x_i=E(x_i), \qquad i=1, \cdots, n,$$

where

$$E(x)=\exp\left\{-n\int^x\sum_{j=1}^{n}\frac{\partial^2M^{(j)}(t, \cdots, t)}{\partial x_p\partial x_q}dt\right\}, \qquad p\neq q.$$

It is shown that $E(x)$ is independent of the choice of distinct values p and q; $p, q=1, \cdots, n$. Further, we have $d\Phi/dx=E(x)$.

E. F. Beckenbach (Los Angeles, Calif.).

Obreškov, N. On the asymptotic behavior of the derivatives of a real function of a real variable. Doklady Akad. Nauk SSSR (N.S.) **67**, 225–228 (1949). (Russian)

Let $\varphi(x)$ be a positive function such that for every positive λ we have $\lim_{x\to\infty}\varphi(\lambda x)/\varphi(x)=\lambda^m$. Let $f(x)$ be a real function such that $f(x)\sim A\varphi(x)$ as $x\to\infty$ and $f^{(n)}(x)>-Mx^{-n}\varphi(x)$ for $x>a$, where $M>0$. Then the author shows that, for $1\leqq i\leqq n-1$,

$$f^{(i)}(x)\sim Am(m-1)\cdots(m-i+1)x^{-i}\varphi(x), \qquad x\to\infty.$$

The same conclusion holds for $i=n$ if $f^{(n)}(x)$ is merely assumed to be monotonic. The analogous result for sequences is also given.

R. P. Boas, Jr. (Providence, R. I.).

Agmon, Shmuel. Sur deux théorèmes de M. S. Mandelbrojt. C. R. Acad. Sci. Paris **228**, 1835–1837 (1949).

Mandelbrojt has given results on approximation on the interval $(0, \infty)$ by polynomials containing only exponents drawn from a sequence $\{\lambda_n\}$ and also on the uniqueness of the solution of the generalized Stieltjes moment problem with exponents λ_n [Ann. Sci. École Norm. Sup. (3) **65**, 101–138 (1948); these Rev. **10**, 436]. The first result was improved by J. Horváth [same C. R. **227**, 1074–1076 (1948); these Rev. **10**, 295]. The author improves both results, the principal improvement consisting in the removal of the restriction that the λ_n are integers. *R. P. Boas, Jr.*

Theory of Sets, Theory of Functions of Real Variables

Denjoy, Arnaud. Quelques propriétés des ensembles rangés. Ann. Soc. Polon. Math. **21** (1948), 187–195 (1949).

The author generalizes the notion of well-ordering as follows. A completely ordered set E is said to be ranked if it is ordinally equivalent to no proper initial segment $E[x; x\varepsilon E, x<a]$ or $E[x; x\varepsilon E, x\leqq a]$ for $a\varepsilon E$. He gives several examples of ranked sets on the line (under the usual ordering) which are not well-ordered, and produces further examples to show that a ranked family of finite well-ordered sets, with the usual ordering for the union, need not be ranked. However, if every set E_γ of a family $\{E_\gamma\}$, $\gamma\varepsilon\Gamma$, is ranked and Γ is well-ordered, then the set $\bigcup E_\gamma$, $\gamma\varepsilon\Gamma$, is ranked. *E. Hewitt* (Seattle, Wash.).

Fraïssé, Roland. Sur une classification des systèmes de relations faisant intervenir les ordinaux transfinis. C. R. Acad. Sci. Paris **228**, 1682–1684 (1949).

There is proposed a pre-ordering of systems of relations of several variables. Then there is defined a well-ordered descending class of equivalence relations among these systems. The extensive distinction of these equivalence relations, and similar propositions, are announced.

R. Arens (Los Angeles, Calif.).

Maximoff, Isaiah. Foundation of transfinite analysis. Tôhoku Math. J. **48**, 292–306 (1941).

Let Ω_i be the first transfinite ordinal number which is preceded by $\aleph_i$ transfinite numbers. Then the author defines a point x of the transfinite space $I_x^{(i)}$ (of order i) by a transfinite sequence $x=\{x_0, x_1, \cdots, x_\alpha, \cdots\}$ of numbers x_α with $0\leqq\alpha<\Omega_i$, $1\leqq x_\alpha<\Omega_i$. The set of all the points x for which $x_{\alpha_0}, x_{\alpha_1}, \cdots, x_{\alpha_\nu}$ are given is called by the author a "portion" or a "group" of order ν with the characteristics $(\alpha_0, \alpha_1, \cdots, \alpha_\nu)$ in $I_x^{(i)}$, according as ν may be finite or not. The set of all the systems $[x^{(1)}, x^{(2)}, \cdots, x^{(m)}]=X$, where every $x^{(s)}$, $1\leqq s\leqq m$, is a point of $I_{x^{(s)}}^{(i)}$, is called a transfinite space $I_X^{(i)}$ of dimension m. The set of all the points X where every $x^{(s)}$, $1\leqq s\leqq m$, belongs to a portion (or group) of $I_{x^{(s)}}^{(i)}$ is called a portion (or group) of $I_X^{(i)}$.

Now let E be any set of points in $I_X^{(i)}$. A point X_0 of $I_X^{(i)}$ is called a "limit (P)-point" for E if every portion containing X_0 and not contained in E contains at least one point X_1 of E different from X_0. A point X_0 of any space I is called the "limit (P)" of a sequence of points $X_1, \cdots, X_\alpha$ $(\alpha<\Omega_i)$ of I if every portion $\pi(X_0)$ in I containing X_0 contains all the points $X_\nu, X_{\nu+1}, \cdots, X_{\nu+\mu}, \cdots$ $(\nu<\Omega_i, \mu<\Omega_i)$, where ν depends on $\pi(X_0)$. Similarly "limit (G)-point" and "limit (G)" are defined if portion is replaced by group. On the basis of these definitions the notions of derivatives of sets, closed and perfect sets, limits of transfinite sequences of sets, Borel sets [cf. also the author's paper, Compositio Math. **7**, 201–213 (1939); these Rev. **1**, 109], and continuous functions are introduced and discussed; cases (P) and (G) are always distinguished according as portions or groups may be used. *A. Rosenthal.*

Maximoff, Isaiah. On a problem of the transfinite analysis. Tôhoku Math. J. **48**, 307–311 (1941).

Employing again the notions and notations of the preceding paper the author calls a function $f(x)$ defined on E $(\subset I_x^{(i)})$ "hypermeasurable (G)" if E contains a perfect (G) set P such that $f(x)$ is continuous (G) on P. If $f(x_1)\neq f(x_2)$ in all cases when $x_1\neq x_2$, $x_1\varepsilon E$, $x_2\varepsilon E$, then the author calls $f(x)$ "regular in E." Let

$$x=\{x_0, x_1, x_2, \cdots, x_\alpha, \cdots\},$$

$\alpha<\Omega_{i+k}$, be a sequence of points x_α of the space $I_x^{(i)}$; then x is called a point of the transfinite space $M_x^{i,k}$. Moreover the author denotes by ρI the point $x\varepsilon M_x^{i,k}$ if

$$x_0=x_1=x_2=\cdots=x_\alpha=\cdots=\rho\varepsilon I_x^{(i)}$$

and the set of all the points ρI is called E_2. He denotes by E_1 the set of all the points x of $M_x^{i,k}$ with the following properties: one of the elements x_α is the point $\{2, 2, \cdots, 2, \cdots\}$ of $I_x^{(i)}$ and every other element is the point $\{1, 1, \cdots, 1, \cdots\}$ of $I_x^{(i)}$. A function $f(x)$ is called a function of Parfentieff if $f(x)$ is "regular" in E_2 and if $E_1\neq f(E_2)$. The author proves that every function $f(x)$ which is hypermeasurable (G) and "regular" in E_2 is a function of Parfentieff. From this he concludes that there is no one-to-one mapping of two sets of powers $2^{\aleph_i}$ and $\aleph_{i+k}$ $(0\leqq i<\Omega_0, 1\leqq k<\Omega_0)$, respectively, by means of hypermeasurable functions. *A. Rosenthal.*

Charles, Fernand. Sur l'extension aux fonctions implicites d'un théorème de Lindelöf et de sa généralisation. Bull. Soc. Roy. Sci. Liége **16**, 254–260 (1947).

In a standard proof of the implicit function theorem for the equation (1) $z=z_0+f(x_1, \cdots, x_n, z)$, with $f(x^0, z_0)=0$, where f satisfies a Lipschitz condition, the first approximation is $z=z_0$. The author shows that an arbitrary continuous function $\phi(x_1, \cdots, x_n)$ can be used as first approximation if $\phi(x^0)=z_0$. He further states that the function z_0 is best as first approximation if the solution of (1) obtained is to exist over a set $|x_i-x_i^0|\leqq h$ $(i=1, \cdots, n)$ with h as large as possible; but his "proof" assumes as necessary an inequality which was only shown to be sufficient in the first part of the paper. *A. B. Brown* (Flushing, N. Y.).

Choquet, Gustave. Application des propriétés descriptives de la fonction "contingent" à la théorie des fonctions de variable réelle et à la géométrie différentielle des variétés cartésiennes. Arch. Math. **1**, 464–467 (1949).

Summary of the author's thesis [J. Math. Pures Appl. (9) **26** (1947), 115–226 (1948); these Rev. **9**, 419].

Moran, P. A. P. On plane sets of fractional dimensions. Proc. London Math. Soc. (2) **51**, 415–423 (1949).

Continuing the problem [Proc. Cambridge Philos. Soc. **42**, 15–23 (1946); these Rev. **7**, 278; Besicovitch and the author, J. London Math. Soc. **20**, 110–120 (1945); these Rev. **8**, 18] of the relation of the α-dimensional measures Λ^α of a linear set A and the Cartesian product C of A and

the interval $(0, h)$, it is shown that

$$\Lambda^{1+\alpha}C \geqq \Gamma(2+\alpha)^{2-\alpha}[\Gamma(1+\tfrac{1}{2}\alpha)]^{-2}h\Lambda^{\alpha}A$$

and this is the best result. Also if C is a plane set whose projection on the x-axis is a measurable set A, where $0<\Lambda^{\alpha}A<\infty$, $0<\alpha\leqq 1$ and with $B(x_0)$ the set of (x_0, y) with x_0 in A, if $\Lambda_*{}^{\beta}B(x_0)>0$ ($\Lambda_*{}^{\beta}$ an outer measure), then $\alpha+\beta\leqq \dim C\leqq 1+\alpha$ and both bounds can be attained.

J. F. Randolph (Rochester, N. Y.).

Brodskiĭ, M. L. On some properties of sets of positive measure. Uspehi Matem. Nauk (N.S.) **4**, no. 3(31), 136–138 (1949). (Russian)

Let x_0 be a point of density of M and y_0 a point of density of N, where M, N are measurable linear sets, and let $f(x, y)$ possess continuous nonvanishing partial derivatives near (x_0, y_0); then, for any z sufficiently near to $z_0=f(x_0, y_0)$ (in particular, when $z=z_0$) there exist x in M and y in N such that $z=f(x, y)$. This theorem is proved and a few natural variants are stated, e.g., extensions to several functions. From the latter it is deduced that if E is the topological product of n measurable linear sets whose measures are positive, then the projection of E on any hyperplane contains an open set which includes the projection of each point of density of E. While this result shows the special nature of such a topological product, it is proved further that, in n-dimensions, any measurable set of positive n-dimensional measure contains the topological product of a measurable set of positive $(n-k)$-dimensional measure in $n-k$ dimensions, together with k completely disconnected one-dimensional sets.

L. C. Young (Madison, Wis.).

Verčenko, I. Ya. On relative differentiability of set functions. Doklady Akad. Nauk SSSR (N.S.) **67**, 417–420 (1949). (Russian)

If K is a circular disc contained in a subset E of the plane and if (x, y) is a point of $E-K$, denote by $P(E, \alpha, (x, y), K)$ the proposition that the tangents from (x, y) to K subtend an angle α and the triangle bounded by these tangents and the segment joining their points of tangency is entirely contained in E. A point (x_0, y_0) is an α point of E if it is the center of some disc K such that whenever $(x, y)\in E-K$, then $P(E, \beta, (x, y), K)$ with $\beta\geqq\alpha$. A sequence $\{E_n\}$ of sets is regular at a point (x_0, y_0) if there exists a positive angle α such that (x_0, y_0) is an α point of E_n for each n. If F and G are arbitrary set functions (defined on the class of all Borel subsets of the plane), the upper [lower] relative derivative of F with respect to G at a point (x, y) is the supremum [infimum] of the set of these numbers c for which there exists a sequence $\{Z_n\}$ of closed sets regular at (x, y), converging to (x, y), and such that $F(Z_n)/G(Z_n)\to c$; if these two numbers are equal, their common value is the relative derivative of F with respect to G. If F_1 and F_2 are Borel measures, write $F_1{}^*$ and $F_2{}^*$, respectively, for the outer measures they induce, $F=F_1-F_2$, and $F^*=F_1{}^*-F_2{}^*$.

The author states (without proof) seven theorems dealing with generalizations of the Vitali covering theorem, differentiation of set functions, and metric density. The following result, stated in terms of the notations established in the preceding paragraph, is typical. If G is a countably additive set function, then there exists a set N such that $F^*(N)=0$ and such that at every point of the complement of N the relative derivative of F with respect to G exists and is finite.

P. R. Halmos (Chicago, Ill.).

Fedorov, V. S. On a property of line integrals. Mat. Sbornik N.S. **24**(66), 15–26 (1949). (Russian)

Let the single-valued real functions $P(x, y)$ and $Q(x, y)$ be continuous in a finite simply connected domain D. For simple closed rectifiable curves L in D, consider the functional $J(L)/\sigma$, where $J(L)=\int_L P\,dx+Q\,dy$, and σ is the area of the finite domain bounded by L. For a given curve L_0 and a given functional $J(L_0)/\sigma$, the author investigates the set E of curves L enclosed by L_0 for which $J(L_0)/\sigma_0=J(L)/\sigma$. He shows that either E consists of all simple closed rectifiable curves enclosed by L_0, so that $J(L)=c\sigma$ where c is the same constant for all L; or for every such L there is in E an infinite sequence $\{L_k\}$ of curves obtained from L by similarity transformations (without rotation) and translations, such that by pairs the L_k enclose nonoverlapping areas, and such that to each L_k there correspond curves L_k' and L_k'' obtained from L_k by translations, with L_k, L_k', L_k'' not overlapping, for which

$$J(L_k')/\sigma_k<J(L_k)/\sigma_k=J(L_0)/\sigma_0<J(L_k'')/\sigma_k.$$

The results still hold if L_0 is replaced by a set of curves bounding a multiply connected domain, or if $J(L)/\sigma$ is replaced by $J(L)/K(L)$, where $K(L)$ is a nonvanishing functional of the same form as $J(L)$. Applications are made in particular to the equation $J(L)=0$, to areal derivatives of functionals $J(L)$, and to a stronger form of Morera's theorem.

E. F. Beckenbach (Los Angeles, Calif.).

Menger, Karl. Stieltjes integrals considered as lengths. Ann. Soc. Polon. Math. **21** (1948), 173–175 (1949).

Necessary and sufficient conditions are given in order that the "length" of the interval $[a, b]$ derived from a "distance" $d(x, y)$, be a left side Stieltjes integral, i.e., the limit of the sums $\sum f(x_i)[g(x_{i+1})-g(x_i)]$. The conditions are (1) $d(x, y)d(y, z)+d(y, x)d(x, z)=d(x, y)d(y, x)$ and (2) $d(x, z)d(y, w)=d(y, z)d(x, w)$ whenever $0=d(x, y)\neq d(x, z)$.

L. C. Young (Madison, Wis.).

***Altwegg, Martin. Ein Modell des Hilbertschen Raumes.** Thesis, Eidgenössische Technische Hochschule, Zürich, 1948. 35 pp.

Following the idea of F. Riesz [Acta Math. **42**, 191–205 (1920)] the author develops the theory of Lebesgue integration without using the notion of Lebesgue measure except that of null sets: a real-valued function $f(x)$ (defined on a subset B of a Euclidean space with a finite measure) is integrable if $f(x)$ is a limit almost everywhere of a sequence of equi-integrable step functions $f_n(x)$, and the integral of $f(x)$ is the limit of the integrals of $f_n(x)$. This method of definition is then shown to be useful in the discussion of properties of concrete L^2-space over B.

S. Kakutani.

Rohlin, V. A. On the fundamental ideas of measure theory. Mat. Sbornik N.S. **25**(67), 107–150 (1949). (Russian)

Proofs of the author's results on decompositions of measure spaces [announced in Doklady Akad. Nauk SSSR (N.S.) **58**, 29–32 (1947); these Rev. **9**, 230].

P. R. Halmos.

Mayrhofer, Karl. Über die Vervollständigung eines volladditiven Inhalts. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. **84**, 95–98 (1947).

The author defines a content to be a finitely additive, nonnegative set function μ, defined on a Boolean algebra $\mathbf{A}$ of sets and such that $\mu(0)=0$. A content μ is called com-

plete if for every set M, with the property that to every $\epsilon>0$ there exist sets E and F in $\mathbf{A}$ such that $E\subset M\subset F$ and $\mu(F-E)<\epsilon$, it is true that $M\epsilon\mathbf{A}$. Every content μ has a minimal complete extension called the completion of μ. The purpose of the note is to prove, without using the known theory of extensions of measures to σ-algebras, that the completion of a countably additive content is countably additive. *P. R. Halmos* (Chicago, Ill.).

Areškin, G. Ya. On a problem concerning the possibility of the inversion of limit and total variation in the theory of completely additive set functions. Uspehi Matem. Nauk (N.S.) **4**, no. 3(31), 134–135 (1949). (Russian)

Let $\{\mu_n\}$ be a sequence of real valued, countably additive set functions defined on a σ-algebra $\mathbf{S}$ of subsets of a set X, and suppose that $\lim_n \mu_n(E)=\mu(E)$ exists for every E in $\mathbf{S}$. Let μ^+, μ^- and $|\mu|$ [or μ_n^+, μ_n^-, and $|\mu_n|$] be the upper, lower, and total variation of μ [or of μ_n], respectively, and let $\{E^+, E^-\}$ be a Hahn decomposition of X with respect to μ (i.e., $E^+\cap E^-=0$, $E^+\cup E^-=X$, and $\mu^+(E^-)=\mu^-(E^+)=0$). Let, finally, $\{E_n^{++}, E_n^{+-}\}$ and $\{E_n^{-+}, E_n^{--}\}$ be Hahn decompositions of E^+ and E^-, respectively, with respect to μ_n. The author proves that a necessary and sufficient condition that $\lim_n |\mu_n|(E)=|\mu|(E)$ for every E in $\mathbf{S}$ is that $\lim_n \mu_n(E_n^{+-})=\lim_n \mu_n(E_n^{-+})=0$. *P. R. Halmos.*

Morse, Marston, and Transue, William. The Fréchet variation and the convergence of multiple Fourier series. Proc. Nat. Acad. Sci. U. S. A. **35**, 395–399 (1949).

For functions $f(x)$ of a single real variable x, the concept of total variation is fundamental. For functions $f(x_1, \cdots, x_n)$ of $n\geqq 2$ real variables, the definitions of the total variation due to Vitali and to Fréchet, respectively, seem to represent plausible generalizations, especially in dealing with multilinear functionals and with multiple Fourier series. The present note is an abstract of results which show that the less restrictive concept of Fréchet variation leads to a set of convergence tests for multiple Fourier series which are stronger than those obtained previously in the literature in terms of the Vitali variation. Further results include (i) general theorems on functions $f(x_1, \cdots, x_n)$ of bounded Fréchet variation which generalize the basic continuity theorems for functions of bounded variation of a single variable, and (ii) a representation theorem for multilinear functionals in terms of a Radon-Stieltjes integral based on Fréchet variation. *T. Radó* (Columbus, Ohio).

Conti, Roberto. Su una nuova classe di funzioni "a variazione limitata" di due variabili e le sue relazioni con le classi H, A, P. Boll. Un. Mat. Ital. (3) **4**, 53–57 (1949).

Let P, H, A, $\bar{A}$ denote the classes of functions $f(x, y)$ of bounded variation on $R: a\leqq x\leqq b$, $c\leqq y\leqq d$, in the senses of Pierpont-Hahn, Hardy-Krause, Arzelà, and in a new sense introduced by the author, respectively: the class A [$\bar{A}$] is defined by the condition that $\sum|f(x_{i+1}, y_{i+1})-f(x_i, y_i)|$ shall be bounded for every finite set of points (x_i, y_i), $i=1, \cdots, n$, of R such that the sequences $\{x_i\}$ and $\{y_i\}$ are monotonic in the same [opposite] senses. The author amplifies known relations of inclusion among P, H, and A [cf. Clarkson and Adams, Trans. Amer. Math. Soc. **35**, 824–854 (1933)] as follows: (i) $H<A\bar{A}<A+\bar{A}<P$, J_w, (ii) $A\bar{A}<J_s$, where J_w [J_s] is the class of functions on R of bounded variation in each variable separately [on every line segment of R]. *H. P. Mulholland* (Beirut).

Kronrod, A. S. On linear and planar variations of functions of several variables. Doklady Akad. Nauk SSSR (N.S.) **66**, 797–800 (1949). (Russian)

The author gives definitions and theorems without proofs. Consider a function $f(\eta)$ continuous on a region J that is either a square or a 2-sphere, and let E_t be the level-set $E[f(\eta)=t]$, M_t the set $E[f(\eta)\leqq t]$, and N_t the set $E[f(\eta)\geqq t]$. A component K of E_t is called regular if it divides J into two domains, and is called a semi-maximum [semi-minimum] component if there is a continuum R such that $R\cdot K\neq 0$, R is not in K, $R\cdot(E_t-K)=0$, and $R\subset M_t$ [$R\subset N_t$]. Let ξ be a fixed point. Suppose that K is a component of E_t, $\xi\overline{\epsilon}K$, and K divides J into two domains J_1 and J_2 with $\xi\epsilon J_1$: then $K^*=J_1+K$ is called the internal characteristic, and $K^{**}=J_2+K$ the external characteristic, of K; K is called a component of increase relative to ξ if K^* is a semi-minimum component for the function $f^*(\eta)$: $f^*(\eta)=f(\eta)$ $(\eta\epsilon J_2)$, $f^*(\eta)=t$ $(\eta\epsilon K^*)$, and a semi-maximum component for $f^{**}(\eta)$: $f^{**}(\eta)=f(\eta)$ $(\eta\epsilon J_1)$, $f^{**}(\eta)=t$ $(\eta\epsilon K^{**})$. Similarly for a component of decrease. Then $f(\eta)$ is called monotone increasing relative to ξ if every regular component K, with $\xi\overline{\epsilon}K$, of every level-set of $f(\eta)$ is a component of increase, and monotone decreasing if $-f(\eta)$ is monotone increasing. Theorem 3: A necessary and sufficient condition for $f(\eta)$ to be monotone increasing [decreasing] is the connectivity of M_t [N_t] for every t.

Let $\Phi_\xi^+(t)$ denote the number of components of E_t that are not components of decrease relative to ξ. Then $V_\xi^+(f)=\int_{-\infty}^{\infty}\Phi_\xi^+(t)dt$ is called the positive variation of $f(\eta)$ from ξ; the similarly defined $V_\xi^-(f)$ is called the negative variation, and $V(f)=V_\xi^+(f)+V_\xi^-(f)$ the linear variation of $f(\eta)$. All the variations introduced are lower semi-continuous as functions of f and invariant under homeomorphic transformations of J onto itself; further [theorem 4] any function of bounded linear variation is the difference of two monotone increasing functions (relative to a given ξ). A function is of bounded linear variation if its derivatives satisfy the Lipschitz condition: apart from this the theory above can be extended to n dimensions.

The planar variation $W(f, G)$ of $f(\eta)$ on an open set $G\subset J$ is defined to be $\int_{-\infty}^{\infty}\nu(G, t)dt$, where $\nu(G, t)$ is the one-dimensional Hausdorff measure of the set $G\cdot E_t$; and for a Borel set M we put $W(f, M)=\inf W(f, G)$ $[G\supset M]$. Theorem 5: for almost all t, $\nu(G, t)$ equals the sum of the lengths of the regular components of E_t. The planar variation $W(f, J)$ is finite if and only if $f(\eta)$ is of bounded variation in Tonelli's sense on J. Theorem 8: if $f(\eta)$ is absolutely continuous (i.e., if $W(f, M)$ is an absolutely continuous function of M) then $W(f, G)=\int_G|\text{grad}_A f(\eta)|dS$, where $\text{grad}_A f(\eta)$ is the asymptotic gradient. The variation $W(f, J)$ is invariant under change of axes, and $W(f, G)$ is a lower semi-continuous function of f. Theorem 11: If the linear and planar variations of $f(\eta)$ on J are both bounded then $f(\eta)$ has a total differential almost everywhere. Apart from theorem 11 these results on planar variation can be extended to n dimensions. [Some of the author's results, especially in his theorems 2, 5, and 8, overlap results of the reviewer [Proc. London Math. Soc. (2) **46**, 290–311 (1940); **50**, 559–560 (1949); especially theorems 1 and 4, and §§ 7, 8, 12; these Rev. **1**, 303; **10**, 520]. The author's planar variation, though differently defined, coincides with the reviewer's total variation.] *H. P. Mulholland* (Beirut).

Nachbin, Leopoldo. Sur les algèbres denses de fonctions différentiables sur une variété. C. R. Acad. Sci. Paris 228, 1549–1551 (1949).

The author establishes the following analogue of the Stone-Weierstrass approximation theorem. Let M be a manifold of class C^r, let A be the algebra of real functions of class C^r on M and let the topology in A be that of uniform convergence on compact sets up to the derivatives of order r. Then a necessary and sufficient condition that a subalgebra F of A be dense in A is that (1) for each x in M there is an f in F with $f(x)\neq 0$, (2) for $x\neq y$ in M there is an f in F with $f(x)\neq f(y)$, (3) for x in M and a tangent vector T at x there is an f in F such that $Tf\neq 0$. *R. Arens.*

Cesari, Lamberto. Sulla trasformazione degli integrali doppi. Ann. Mat. Pura Appl. (4) 27, 321–374 (1948).

Let $\Phi: x=x(u,v),\ y=y(u,v)$, be a continuous plane transformation defined on the unit square $A: 0\leqq u, v\leqq 1$. Denote the generalized Jacobian of the transformation Φ by $H(u,v)$ and let $\Psi(x,y;\Phi)$ and $n(x,y;\Phi)$ denote the absolute and relative characteristic (counting) functions for the transformation Φ. The author proves that if Φ is absolutely continuous, then

$$\int\int_A f(x(u,v),y(u,v))\,|H(u,v)|\,dudv = \int\int_{\Phi(A)} f(x,y)\Psi(x,y;\Phi)\,dxdy,$$

$$\int\int_A f(x(u,v),y(u,v))H(u,v)\,dudv = \int\int_{\Phi(A)} f(x,y)n(x,y)\,dxdy,$$

whenever at least one of the functions $f(x(u,v),y(u,v))H(u,v)$, $f(x,y)\Psi(x,y;\Phi)$ is summable. The fundamental idea behind these formulas is the replacement of the ordinary counting function for the transformation Φ by a refined counting function which possesses certain continuity properties. The Banach concept of absolute continuity is then suitably modified to fit the new multiplicity function. A program based on these same general ideas was initiated by T. Radó in 1928. The resulting theory, as well as an account of its literature, is presented in part IV of his book [Length and Area, Amer. Math. Soc. Colloquium Publ., v. 30, New York, 1948; these Rev. 9, 505]. Comparison of the work of Cesari with this theory leads to some interesting questions. In particular, the relationship between the generalized Jacobians used in the two theories seems to deserve further study. *R. G. Helsel* (Columbus, Ohio).

Reichelderfer, Paul V. Law of transformation for essential generalized Jacobians. Duke Math. J. 16, 73–83 (1949).

Given two systems of Cartesian coordinates $x_1x_2x_3$, $x_1'x_2'x_3'$ similarly oriented in a Euclidean 3-space E_3, the relations (1) $x_i'=c_i+\sum_{j=1}^{3}a_{ij}x_j$, $i=1,2,3$, hold, where $\|a_{ij}\|$ is normal and orthogonal, $|a_{ij}|=1$. Given a Fréchet surface S in E_3 and a representation of S, $x_i=x_i(u,v)$, $i=1,2,3$, $(u,v)\in R$, R a bounded simple Jordan region in the (u,v)-plane, then the relations (1) give a further representation of S, $x_i'=x_i'(u,v)$, $i=1,2,3$, $(u,v)\in R$. If S has finite Lebesgue area, then (in several ways) it is possible to introduce notions of generalized Jacobians $J_i(u,v)$, $J_i'(u,v)$, $i=1,2,3$, of the couples x_2x_3, $x_2'x_3'$, x_3x_1, $\cdots$, $x_1'x_2'$ of functions $x_i(u,v)$, $x_i'(u,v)$. These generalized Jacobians exist almost everywhere in R^0, R^0 being the set of the interior points of R [see T. Radó, Length and Area, Amer. Math. Soc. Colloquium Publ., v. 30, New York, 1948; L. Cesari, Atti Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 13, 1323–1481 (1943), in particular, p. 1432; Ann. Mat. Pura Appl. (4) 27, 321–374 (1948), in particular, p. 330; these Rev. 9, 509; 8, 142; see also the preceding review]. Utilizing the generalized Jacobians of Radó the author proves that, for each surface S of finite Lebesgue area, the classical relations

$$J_i'(u,v)=a_{i0}J_1(u,v)+a_{i2}J_2(u,v)+a_{i3}J_3(u,v),\qquad i=1,2,3,$$

hold almost everywhere in R^0. An analogous result, under the same conditions, has been obtained by the reviewer for the generalized Jacobians of Cesari together with other tangential properties [Comment. Math. Helv. 22 (1949), 1–16 (1948); these Rev. 10, 109]. *L. Cesari* (Bologna).

Theory of Functions of Complex Variables

***Heffter, Lothar. Kurvenintegrale und Begründung der Funktionentheorie.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1948. iv+48 pp. 5.40 DM.

In the present work there is presented an orderly development of some of the fundamentals of the theory of analytic functions of a complex variable. The first five chapters deal respectively with functions of a real variable, rectifiable plane curves, curvilinear integrals, Stieltjes integrals, and conditions for independence of path in real integration. The above development leads to the establishment, in the sixth chapter, of integration theorems for analytic functions of a complex variable, including theorems of Cauchy, Goursat, and Morera. It is emphasized that the consequent mean-value characterizations of analytic functions $f(z)$ do not presuppose anything concerning the derivative $f'(z)$. A seventh chapter is devoted to remarks concerning the papers of a selective bibliography. *E. F. Beckenbach.*

Mac Lane, Gerald R. Polynomials with zeros on a rectifiable Jordan curve. Duke Math. J. 16, 461–477 (1949).

The following theorem is proved. If $f(z)$ is analytic and nowhere zero in a finite simply-connected region R whose boundary is a rectifiable Jordan curve K, then there exists a sequence of polynomials, all zeros of which lie on K, converging uniformly to $f(z)$ on any closed subset of R. Such a sequence of polynomials is actually exhibited. It is shown, moreover, that the zeros must be everywhere dense on K. The region R can be multiply-connected if the polynomials are replaced by rational functions with poles outside R. *E. N. Nilson* (Hartford, Conn.).

Zygmund, A. Errata to the paper "On a theorem of Hadamard." Ann. Soc. Polon. Math. 21 (1948), 357–358 (1949).

The paper appeared in the same vol., 52–69 (1948); these Rev. 10, 186.

Blambert, Maurice. Sur le problème de Hadamard-Mandelbrojt ou problème de la composition des singularités des séries de Dirichlet générales. C. R. Acad. Sci. Paris 228, 1915–1916 (1949).

Continuing a previous note [same C. R. 226, 1666–1668 (1948); these Rev. 9, 576], the author announces two theorems generalizing some results of G. Pólya [Ann. of Math. (2) 34, 731–777 (1933)] about the singularities of the Hadamard product of power series to the Hadamard-

Mandelbrojt product of Dirichlet series. The precise statements are too long to be reproduced here.

A. Dvoretzky (Princeton, N. J.).

Teghem, J. Sur les conditions d'applicabilité d'une méthode de prolongement analytique de Borel. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 177–185 (1949).

Borel's method of analytic continuation may be described thus. Let $f_n(z)=\sum c_\nu^{(n)}z^\nu$ with $\lim f_n(z)=1/(1-z)$, uniformly in a domain A. Let $F(z)=\sum u_\nu z^\nu$ be analytic inside a contour $\mathfrak{C}$, and $\mathfrak{A}=\bigcap_{z\in\mathfrak{C}}xA$. Let $F_n(z)=\sum c_\nu^{(n)}u_\nu z^\nu$. Then, in $\mathfrak{A}$, $\lim F_n(z)=F(z)$. This is known to hold if A contains the origin. The author shows that it also holds in the contrary case, at least for certain classes of well behaved functions F which are regular in the exterior of a bounded domain, and have at worst a pole at infinity whose order is somewhat restricted by the sequence $\{f_n\}$. Several examples are given, one of which is equivalent to a method of summation based on a transformation of Euler discussed in the author's thesis [Université Libre de Bruxelles, 1946; these Rev. **10**, 112].

R. C. Buck (Madison, Wis.).

Germay, R.-H.-J. Sur une proposition de la théorie générale des fonctions analytiques. Bull. Soc. Roy. Sci. Liége **17**, 62–65 (1948).

The author constructs an analytic function having singular points at an arbitrary sequence b_n, $|b_n|\to\infty$, with prescribed principal parts, and at the same time having prescribed values for itself and its first μ_n derivatives at another arbitrary sequence a_n, $|a_n|\to\infty$.

R. P. Boas, Jr.

Germay, R.-H.-J. Sur une application d'un théorème de E. Picard relatif aux produits indéfinis de facteurs primaires. Bull. Soc. Roy. Sci. Liége **17**, 138–143 (1948).

Let $\{a_n\}$ be a sequence of points not on a circle C of radius R but with $|a_n|\to R$. The author constructs a function analytic except on C with prescribed values for itself and its first μ_n derivatives at a_n.

R. P. Boas, Jr.

Germay, R.-H. Extension d'un théorème d'E. Picard sur les produits indéfinis de facteurs primaires. Bull. Soc. Roy. Sci. Liége **17**, 180–185 (1948).

The author extends the construction described in the preceding review to the case of two circles and two corresponding sequences, the desired function being regular between the circles.

R. P. Boas, Jr. (Providence, R. I.).

Germay, R.-H.-J. Un exemple de produit indéfini de facteurs primaires dont les zéros sont les racines d'équations non résolues. Bull. Soc. Roy. Sci. Liége **17**, 224–229 (1948).

The author shows that the (unique, real, positive) root in $|z|<1$ of $ze^{\lambda-z}=1$ is a monotonic and continuous function of λ. Using this fact he shows how to construct an analytic function whose zeros are the roots ξ_n of this equation corresponding to a given sequence of numbers $\lambda_n>1$.

R. P. Boas, Jr. (Providence, R. I.).

Germay, R.-H.-J. Un exemple de produit indéfini de facteurs primaires dont les zéros sont les racines d'équations non résolues. II. Bull. Soc. Roy. Sci. Liége **17**, 294–296 (1948).

A generalization of the construction reviewed above.

R. P. Boas, Jr. (Providence, R. I.).

Germay, R.-H.-J. Remarque sur une application du théorème de Rouché. Bull. Soc. Roy. Sci. Liége **17**, 303–307 (1948).

The author discusses the roots of $z^n e^{\lambda-z}=1$.

R. P. Boas, Jr. (Providence, R. I.).

Kufarev, P. P. On a system of differential equations. Tomsk. Gos. Univ. Učenye Zapiski **1948**, no. 8, 61–72 (1948). (Russian)

Let $G(t)$, $t_0\leqq t<T$, be a one-parameter family of bounded schlicht domains, $G(t'')\subset G(t')$, $t''>t'$, which are obtained from some given initial polygonal domain $G(t_0)$ by prolonging a rectilinear slit $L(t)$ into the interior of $G(t_0)$. The origin $z=0$ is assumed to lie in the interior of each domain $G(t)$, and the function which maps $|w|<1$ onto $G(t)$ is denoted by $z=\Phi(w,t)$ ($\Phi(0,t)=0$, $\Phi_w'(0,t)>0$). The author applies the method of Löwner to obtain differential equations for the parameters which occur in the Schwarz-Christoffel formula for $\Phi(w,t)$.

D. C. Spencer.

Garabedian, P. R. Distortion of length in conformal mapping. Duke Math. J. **16**, 439–459 (1949).

This paper is concerned with the following problem. Let D be a finite domain in the complex plane bounded by n analytic curves $C_1, C_2, \cdots, C_n$. Among all analytic functions $f(z)$ which are regular and single-valued in D, except for a simple pole of residue 1 at $z=\zeta$ ($\zeta\in D$), and such that $f'(z)$ is continuous on C, to find the particular function $f_0(z)$ which minimizes $\int_C|f'(z)|ds$ ($ds=|dz|$). The author shows that this problem is closely related to another extremal problem, namely: let $F(z)$ be regular and single-valued in D, $F(\zeta)=0$, $F'(\zeta)\geqq 0$, and let $\lim_{z\to C}|F(z)-b_\nu|\leqq 1$, where the b_ν are constants depending on $F(z)$; to find the function $F_0(z)$ which maximizes $F'(\zeta)$. It is shown that $|F_0(z)-b_\nu|=1$, $z\in C_\nu$, where the b_ν are suitably chosen constants, and that $f_0(z)$ and $F_0(z)$ are related by the inequality (a) $i\{F_0(z)-b_\nu\}f_0'(z)dz\geqq 0$ ($z\in C_\nu$, $\nu=1,\cdots,n$); furthermore, $2\pi F_0'(\zeta)=\int_C|f_0'(z)|ds$. The inequality (a), which generalizes an inequality in the theory of bounded functions established by the author in his thesis [Harvard, 1948] is the main result of the paper; the extremal properties of both $f_0(z)$ and $F_0(z)$ are easily deduced from (a) by a suitable application of the residue theorem. In order to establish the existence of two functions $f_0(z)$ and $F_0(z)$ which satisfy (a), the author develops a theory of variations of branch-points of certain classes of Riemann surfaces, these variations being such as to leave the conformal type of these surfaces invariant. This variational theory is applied to the extremal problem solved by $F_0(z)$ and leads finally to (a). Several other problems to which this variation of branch-points might be successfully applied are also pointed out.

Z. Nehari (St. Louis, Mo.).

Koebe, Paul. Zur allgemeinen Iterationstheorie der Uniformisierung algebraischer Funktionen. Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **93**, 43–66 (1941).

This paper is concerned with the existence theorems for the uniformization quantities associated with a given closed Riemann surface. The development is based on the so-called iteration method. The present paper constitutes a continuation of two previous papers [Ber. Verh. Sächs. Akad. Wiss. Leipzig Math.-Phys. Kl. **89**, 173–204 (1937); **91**, 135–192 (1939); these Rev. **1**, 114] concerned with this method.

M. Heins (Providence, R. I.).

van der Waerden, B. L. Topologie und Uniformisierung der Riemannschen Flächen. Ber. Verh. Sachs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **93**, 147–160 (1941).

An exposition of the uniformization theorem for triangulated Riemann surfaces. The essential novelty lies in simplifications of the topological part of the proof (exhaustion of a simply-connected surface by elementary surfaces).

M. Heins (Providence, R. I.).

Hayman, W. K. Inequalities in the theory of functions. Proc. London Math. Soc. (2) **51**, 450–473 (1949).

It is assumed that a function $f(z)$, analytic in $|z|<1$, omits a sequence of values w_n, $|w_n|=r_n$. Under certain regularity conditions the author proves that the maximum modulus $M(\rho, f)=O((1-\rho)^{-2})$. Assuming that $r_n<r_{n+1}$ and setting $d_n=\log r_{n+1}/r_n$, the condition is $\sum d_n^2<\infty$, and if the w_n are real, $w_n<0$, this condition is also necessary. These results are actually proved in the form of more precise inequalities. By the principle of subordination it is sufficient to prove the principal theorem for the function $\varphi(z)$ which maps $|z|<1$ onto the universal covering surface of the plane, deleted at $\{w_n\}_1^\infty$. The author considers a sequence of fundamental regions and the mapping by $\log \varphi(z)$. To this situation the method of Grötzsch-Ahlfors applies and ultimately leads to the result.

L. Ahlfors.

Thiem, Le-Van. Über das Umkehrproblem der Wertverteilungslehre. Comment. Math. Helv. **23**, 26–49 (1949).

The author partially solves the inverse problem in the value distribution theory for meromorphic functions [see R. Nevanlinna, Eindeutige Analytische Funktionen, Springer, Berlin, 1936, pp. 256, 292; also E. Ullrich, Nachr. Ges. Wiss. Göttingen. Fachgruppe I. (N.S.) **1**, 135–150 (1936)]. For Riemann surfaces W with a finite number of periodic ends [Ullrich, loc. cit.], the author defines a quantity Δ which he calls the characteristic of W. He then proves the following three theorems. (A) The function generating a surface W with m periodic ends is of mean type of order $\lambda=\frac{1}{2}m(1+\Delta^2/(m^2\pi^2))$. If $\Delta=0$, the defects $\delta(a_\nu)$ and the indices $\epsilon(a_\nu)$ are rational. (B) If we prescribe the points a_ν and associated numbers δ_ν and ϵ_ν, $\nu=1, 2, \cdots, q$, $0\leqq\delta_\nu\leqq 1$ and $0\leqq\epsilon_\nu\leqq 1$, such that (1) $\sum_{\nu=1}^q\delta_\nu+\sum_{\nu=1}^q\epsilon_\nu=2$; (2) δ_ν, ϵ_ν are rational; (3) $\sum_{\nu=1}^q\delta_\nu\neq 0$; (4) $\delta_\nu+\epsilon_\nu\leqq 1$, and <1 if $\epsilon_\nu\neq 0$, then the inverse problem has a solution in the class of functions generating the surfaces W with $\Delta=0$; that is, there exists a function in this class which has the defects $\delta(a_\nu)=\delta_\nu$ and indices $\epsilon(a_\nu)=\epsilon_\nu$. (C) The surface W solving (B) can always be so selected that it is branched over at most three points. In proving these theorems, the author employs the theory of quasi-conformal mapping.

G. Springer.

Dinghas, Alexander. Zur Abschätzung der a-Stellen ganzer transzendenter Funktionen mit Hilfe der Shimizu-Ahlforsschen Charakteristik. Math. Ann. **120**, 581–584 (1949).

For entire functions the author gives comparisons between $n(r, a)$ and $S(r)$, the area of the image surface on the sphere. Simple estimates are obtained, notably under the condition $\int^\infty \mu^2 r^{-1}dr=\infty$, where $\mu=\mu(r, a)=\min(1, \min|w(z)-a|)$ for $|z|=r$.

L. Ahlfors (Cambridge, Mass.).

Rådström, Hans. On the zeros of successive derivatives of entire functions. Proc. Nat. Acad. Sci. U. S. A. **35**, 399–404 (1949).

The author proves that if $f(z)=\sum b_n z^n$ is any entire function of order greater than one, then there exist numbers ω_n, $|\omega_n|=1$, such that the function $g(z)=\sum\omega_n b_n z^n$ has the property that zero is a limit point of the zeros of the successive derivatives $g^{(n)}(z)$. This is not true for the function $f(z)=e^z$ of order one. Some partial results are appended in which the modifying factors ω_n are allowed to take only the values 1 or -1.

R. C. Buck (Madison, Wis.).

Bernštein, S. N. Functions of finite degree and functions of finite semidegree. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 111–124 (1949). (Russian)

A function $G_p(z)$ of finite degree p is what is usually called (in English) a function of exponential type p. A function $H_p(z)$ of finite semidegree p is defined here by $H_p(z)=\sum a_k z^k/(2k)!$ with $\limsup|a_k|^{1/(2k)}\leqq p$, so that $H_p(z^2)$ is an even $G_p(z)$. By using this representation and earlier results of his own on polynomial approximation [cf. in particular C. R. (Doklady) Acad. Sci. URSS (N.S.) **54**, 475–478 (1946); these Rev. **8**, 509] the author obtains several results about functions of finite semidegree and approximation by means of them; he then uses them to solve some new problems about functions of finite degree.

Some typical results are as follows. Let $A_p^*f(x)$ denote the best approximation to $f(x)$ for $x\geqq 0$ by functions $H_p(x)$; $E_n(f(x); a, b)$, the best approximation to $f(x)$ on (a, b) by polynomials of degree n. Then if

$$|f(x)|\leqq H^*(x)=c\prod_{n=1}^{\infty}|1+x/\alpha_n^2|,$$

$x\geqq 0$, where $\sum 1/|\alpha_n|<\infty$, we have, for $q>0$,

$$\lim_{m\to\infty}E_m\left(f(x); 0, \left(\frac{2m}{q\pm 0}\right)^2\right)=A_{q\pm 0}^*f(x).$$

Hence a function satisfying $f(x)\leqq H^*(x)$ is an entire function of semidegree p if and only if for every $\alpha>0$ there is a polynomial $P_m(x)$ of degree m such that for any positive but for no negative ϵ the inequality $|f(x)-P_m(x)|<\alpha$ holds for $0\leqq x\leqq[2m/(p+\epsilon)]^2$. A further corollary is that if $P_n(x)$ are polynomials which on $(0, \lambda_n)$ satisfy $|P_n(x)|\leqq H^*(x)$, where $\liminf 2n/\lambda_n^{\frac{1}{2}}=p$, then a subsequence $P_{n_k}(x)$ converges to an $H_p(x)$. Using this corollary and known results about polynomials the author proves that, if $H_p(x)\leqq M$ for $x\geqq 0$, then for $b>0$ we have

$$|H_p^{(k)}(-b)|\leqq M(d/dx)^k\cosh(px^{\frac{1}{2}})|_{x=b}=L_p^{(k)};$$

equality is attained for $M\cos(px^{\frac{1}{2}})$.

The author solves the following three problems. (1) Find the $G_p(x)$ with $G_p(\xi)=M>0$, $-b<\xi<b$, deviating least from zero outside $(-b, b)$. Solution: $G_p(x)=L_p\cos p(x^2-b^2)^{\frac{1}{2}}$, where $L_p=M/\cosh[p(b^2-\xi^2)^{\frac{1}{2}}]$. Corollary: if $|G_p(x)|\leqq L$ outside $(-b, b)$, then $|G_p(x)|\leqq L\cosh(pb)$ on the whole real axis, with equality for $L\cos p(x^2-b^2)^{\frac{1}{2}}$ at $x=0$. (2) Find the $G_p(x)$, with $G_p^{(m)}(0)=a_m$ assigned for a certain m, deviating least from zero outside $(-b, b)$. For even m the solution follows from the theorem quoted at the end of the preceding paragraph; for odd m the solution is trivial (a multiple of $\sin px$) if $b\leqq\frac{1}{2}\pi/p$, but if $b>\frac{1}{2}\pi/p$ the solution involves elliptic functions. (3) Find an even $G_p(x)$ (now of degree not exceeding p) with given values $G_p(0)=a_0$, $G_p''(0)=a_2$, deviating least from 0 on the real axis. Solution: $a_0\cos p_1 t$ if $0\leqq -a_2/a_0=p_1^2\leqq p^2$; otherwise $M\cos(p^2t^2+c^2)^{\frac{1}{2}}$, where $M\cos c=a_0$, $-Mp^2c^{-1}\sin c=a_2$, and $c<\pi$ satisfies $p^2\tan c+ca_2/a_0=0$. One verifies directly that this solution satisfies the prescribed conditions; if another function satisfied them and deviated less from zero, the difference would

be an even bounded function $F_p(t)$ such that

$$(-1)^k F_p(c_k/p) = A_k > 0$$

($k=1, 2, \cdots$), $c_k^2 = k^2\pi^2 - c^2$, while $F_p(0) = F_p''(0)$. Now such an $F_p(t)$ is represented by the interpolation formula

$$(p^2t^2+c^2)^{\frac{1}{2}} \sin (p^2t^2+c^2)^{\frac{1}{2}} \left\{ \frac{F_p(ic/p)}{p^2t^2+c^2} + 2\sum_{k=1}^{\infty} \frac{(-1)^k F_p(c_k/p)}{c_k^2 - p^2t^2} \right\},$$

and computation of $F_p(0)$ and $F_p''(0)$ from it leads to a contradiction. *R. P. Boas, Jr.* (Providence, R. I.).

Bernštein, S. N. On majorants of finite or quasi-finite growth. Doklady Akad. Nauk SSSR (N.S.) 65, 117–120 (1949). (Russian)

Let $G_p(x)$ be a generic notation for entire functions of finite degree p [also known as functions of exponential type p]. The author calls $H(x) \geqq 0$ a majorant (or function) of finite growth, and writes $H(x) \epsilon \mathfrak{M}$, if the inequality (*) $|G_p(x)| \leqq H(x)$, $-\infty < x < \infty$, implies $|G_p^{(k)}(x)| \leqq H_{k,p}(x)$, $-\infty < x < \infty$, where (*) $\limsup_{k\to\infty} \{H_{k,p}(x)\}^{1/k} \leqq p$ for fixed x. [In an earlier paper, same Doklady (N.S.) 60, 949–952 (1948); these Rev. 10, 29, he gave two characterizations of majorants of finite growth, one of which was that (*) implies the existence of an entire $H_p(z)$ of degree p such that $|G_p(z)| \leqq |H_p(z)|$ for all complex z.] He also introduces the class $\mathfrak{M}^*$ of majorants of quasi-finite growth, obtained if p is replaced by some p^* in (*). However, as he points out in the paper reviewed below, B. Levin proved independently [see the second following review] that actually $\mathfrak{M}^* = \mathfrak{M}$ for majorants which are the absolute values of entire functions of finite degree, and it is with such majorants that the present paper is chiefly concerned. Proofs are omitted. Some typical results are as follows.

Theorem 1. If α is real and $H_0(x)$ is an entire function of genus zero, then $e^{\alpha x}|H_0(x)| \epsilon \mathfrak{M}^*$. Theorem 2. If $H(x)$ is the absolute value of an entire function $f(z)$, a necessary and sufficient condition for $H(x) \epsilon \mathfrak{M}^*$ is that (†) $\sum |\beta_k|/(\alpha_k^2+\beta_k^2) < \infty$, where $\alpha_k + i\beta_k$ are the zeros of $f(z)$. [The statement given in the paper appears to contain a redundant condition. The form quoted here is taken from a footnote in the author's paper reviewed below, where the necessity is proved, and it is pointed out that the sufficiency (and a sharper conclusion) follow from Levin's results: see the second following review.] Corollary 2. If $f(z)$ is an entire function of finite degree with no zeros in the upper half plane, $|f(x)| \epsilon \mathfrak{M}^*$. Corollary 3. If $G_p(x) \geqq 0$ then $G_p \epsilon \mathfrak{M}^*$ if and only if $G_p(x) = |G_{p/2}(x)|^2$. Corollary 4. An entire function $H(x) \geqq 0$ of finite degree such that

$$\sup_R \int_1^R x^2 \log |H(x)H(-x)|\, dx < \infty$$

belongs to $\mathfrak{M}^*$. [The last two corollaries follow from results of Ahiezer, same Doklady (N.S.) 63, 475–478 (1948); these Rev. 10, 289.]

Call a majorant $H(x)$ additive if $H(x)+H(-x) \epsilon \mathfrak{M}$. Corollary 6. If $H_1(x)$ and $H_2(x)$ are absolute values of functions of finite degree and belong to $\mathfrak{M}^*$, then $H_1+H_2 \epsilon \mathfrak{M}^*$ if and only if both H_1 and H_2 are additive; this is true in particular if $\inf_x H_j(x) \geqq c > 0$, $j=1, 2$. If $H(x) \geqq c > 0$, call $H(x)$ almost increasing if there is a constant $k \geqq 1$ such that for $\lambda \geqq 1$ we have $H(x) \leqq kH(\lambda x)$. Theorem 3. If $H(x)$ is the absolute value of a function of finite degree, belongs to $\mathfrak{M}^*$ and is almost increasing, then $\lim_{x\to\pm\infty} e^{-a|x|} H(x) = 0$ for all $a > 0$. Call $H(x)$ an antimajorant if the inequality (*) is compatible for an aggregate of functions $G_p(x)$ of a given positive degree p with the property $\sup |G_p(x)| = \infty$ on some real interval (for example, e^x and 1 are majorants but e^x+1 is an antimajorant); a necessary condition follows from corollary 4. Theorem 4. If $H(x)$ is an almost increasing entire function of finite degree then the failure of (†) is necessary and sufficient for H to be an antimajorant, and also in order that every continuous function $f(x)$ which is $o(H(x))$ can be approximated by functions $G_p(x)$ with a fixed p in the sense that for every $\epsilon > 0$ there is a $G_p(x)$ such that $|f(x) - G_p(x)| < \epsilon H(x)$. *R. P. Boas, Jr.*

Bernštein, S. N. On additive majorants of finite growth. Doklady Akad. Nauk SSSR (N.S.) 66, 545–548 (1949). (Russian)

For the definition of the class $\mathfrak{M}$ of majorants see the preceding review; departing from the notation of the paper reviewed there, the author now calls a majorant $H(x)$ additive ($H(x) \epsilon \mathfrak{A}$) if for some positive c the function $H(x)+cx \epsilon \mathfrak{M}$; furthermore, he restricts himself to those majorants which are absolute values of entire functions of finite degree. He begins by showing, by the use of corollary 4 of the preceding paper, that if H and H_1 belong to $\mathfrak{A}$ then so does $H+H_1$. Corollaries: (1) if $H(x) \epsilon \mathfrak{A}$ and $|f(x)| \leqq H(x)$ then $|f(x)| \epsilon \mathfrak{A}$; (2) if $H(x) \epsilon \mathfrak{M}$ and $H(x) > c > 0$, then $H(x) \epsilon \mathfrak{A}$; (3) if $H(x)$ is an even majorant, then the condition $\int_{-\infty}^{\infty} (1+x^2)^{-1} \log^+ H(x)\, dx$ is necessary and sufficient for $H(x) \epsilon \mathfrak{A}$.

Call $H(x)$ canonical if

$$H(z) = ae^{ibz} z^m \prod_{k=1}^{\infty} \left(1 - \frac{z}{\alpha_k + i\beta_k}\right) e^{\alpha_k z/(\alpha_k^2+\beta_k^2)}$$

with b real; except for the condition on b a necessary and sufficient condition is (†) of the preceding review [cf. Ahiezer, loc. cit. in the preceding review]. Lemma: two canonical functions $f_1(z)$ and $f_2(z)$ of the same degree, having a common majorant $H(x)$, have the same indicator diagram. [The author notes that this is essentially contained in Levin's paper reviewed below.] Theorem 2 is a special case of theorem 3 of the paper reviewed above, but is given a simplified proof here: if $F(x) \epsilon \mathfrak{A}$ is canonical of degree p its indicator diagram consists of the segment $(-pi, pi)$ of the imaginary axis.

Definitions. Let $F(z)$ have an indicator diagram which is a segment parallel to the imaginary axis and suppose that for every $\epsilon > 0$ we can write $F(z) = F_1(z)H_0(z)$, where $H_0(z)$ is of degree zero and $\lim_{r\to\infty} r^{-1} \log |F_1(re^{i\theta})| = \delta_F(\theta)$ exists for all θ with $|\sin\theta| \geqq \epsilon$; then we call $\delta_F(\theta)$ the degree index of $F(z)$ in the direction θ. Call a majorant $F(x) \geqq 0$ almost positive if there is a $G_0(x)$ of degree 0 ($G_0(x) \not\equiv 0$) such that $|G_0(x)| \leqq F(x)$ for all real x (if $F(x) > c > 0$ it is almost positive). Lemma: every almost positive majorant is the absolute value of a canonical $f(x)$ of finite degree p, having $\delta_f(\theta) = p\,|\sin\theta|$ in every direction. From this lemma it is deduced that [theorem 3] the degree of $G_p(z)G_q(z)$ is $p+q$ if G_p and G_q are canonical functions. In a footnote the author discusses the connection between the results of his preceding paper and the paper of Levin reviewed below.

R. P. Boas, Jr. (Providence, R. I.).

Levin, B. Ya. On some extremal properties of entire functions of finite degree. Doklady Akad. Nauk SSSR (N.S.) 65, 605–608 (1949). (Russian)

The author generalizes a number of extremal properties previously obtained by various authors. His work is based on the following lemma. Let $\omega(z)$ and $f(z)$ be entire functions of exponential type, $\omega(z)$ having no zeros in the lower half plane, and let $|f(x)| \leqq |\omega(x)|$ for all real x; then

$|f(z)/\omega(z)| \leqq e^{k|y|}$ for $y \leqq 0$; and $h_f(\theta) = h_\omega(\theta) + k|\sin\theta|$ $(\pi \leqq \theta \leqq 2\pi)$, where $h_f(\theta)$ is the indicator of $f(z)$. Corollary [credited to M. Krein]: the indicator diagram of $\omega(z)$ is symmetric with respect to a line parallel to the real axis. The following definitions are introduced. The "defect" of $\omega(z)$ is $d_\omega = \frac{1}{2}[h_\omega(-\pi/2) - h_\omega(\pi/2)]$. Class P consists of entire functions of exponential type with negative defect and no roots in the lower half plane; P_σ consists of functions of type σ which belong to P. An additive homogeneous operator which carries a function of class P into a function of class P is called a $\mathfrak{B}$-operator. Lemma: if $f(z)$ and $\omega(z)$ are entire functions of exponential type and the type of $\omega(z)$ is not less than that of $f(z)$, then $f(z) - u\omega(z)$ is of class P for every u with $|u| \geqq 1$ if and only if $\omega(z) \epsilon P$ and $|f(x)| \leqq |\omega(x)|$ for all real x.

Theorem 1. For any $\mathfrak{B}$-operator $\mathfrak{B}$, if $\omega(z) \epsilon P_\sigma$ and $f(z)$ is of exponential type $\tau \leqq \sigma$, then $|f(x)| \leqq |\omega(x)|$ for all real x implies $|\mathfrak{B}f(x)| \leqq |\mathfrak{B}\omega(x)|$ for all real x. Theorem 2. If $\omega(z) = \sum b_k z^k$ and $f(z) = \sum a_k z^k$ are as in theorem 1, and $|f(x)| \leqq |\omega(x)|$, then $|\sum_{k=0}^{n} \mu_k^{(n)} b_k x^k| \leqq |\sum_{k=0}^{n} \mu_k^{(n)} a_k z^k|$, where $\mu_k^{(n)} = \prod_{j=1}^{k-1}(1 - j/n)$, $k = 1, 2, \cdots, n$ $(\mu_0 = 1)$; note that the polynomials in this inequality converge to the original functions as $n \to \infty$. By combining the second lemma quoted above with results given in his earlier papers [C. R. (Doklady) Acad. Sci. URSS (N.S.) 41, 47–50, 99–100 (1943); these Rev. 6, 59, 60] and stated here as theorem 3 in a formulation due to M. Krein, the author derives the following results. Let $\omega(z) \epsilon P_\sigma$ and $\omega(x)/\tilde{\omega}(x) = e^{i\theta(x)}$. Theorem 4. Let $f(x)$ be a real entire function of exponential type $\tau \leqq \sigma$ and let $\psi(\theta) = f(x)/|\omega(x)|$ be bounded for real x. Then for $|h| \leqq \pi$,

$$|\psi(\theta + \tfrac{1}{2}h) - \psi(\theta - \tfrac{1}{2}h)| \leqq 2|\sin \tfrac{1}{2}h| \sup_{-\infty < x < \infty} |\psi(\theta)|.$$

Theorem 5. For $|h| \leqq 2\pi$,

$$\sup_{-\infty < \theta < \infty} \left| \int_{\theta - \frac{1}{2}h}^{\theta + \frac{1}{2}h} \psi(\theta) d\theta \right| \geqq 2|\sin \tfrac{1}{2}h| \sup_{-\infty < \theta < \infty} |\psi(\theta)|.$$

For $\omega(x) = e^{i\sigma x}$ theorems 4 and 5 reduce to known results [see Bernstein, same Doklady (N.S.) 60, 1487–1490 (1948); these Rev. 9, 579]. *R. P. Boas, Jr.* (Providence, R. I.).

Al'per, S. Ya. On the completeness of systems of analytic functions. Doklady Akad. Nauk SSSR (N.S.) 66, 1029–1032 (1949). (Russian)

The author proves the following theorem. Let the functions $g_n(z) = \sum_{k=0}^{n} b_k^{(n)} z^k$ be a complete set of analytic functions in a simply-connected region G containing the origin. Let $f(z) = \sum_{k=0}^{\infty} a_k z^k$, with all $a_k \neq 0$, be analytic in a neighborhood of the origin. [These conditions on $f(z)$ will be retained throughout this review.] Then the system $F_n(z) = \sum_{k=0}^{n} a_k b_k^{(n)} z^k$ is complete in any simply-connected region containing neither $z = \infty$ nor any point $\alpha\beta$, where α is a boundary point of G and β is a singular point of $f(z)$. The author also states the following result: if furthermore the $g_n(z)$ form a basis in $|z| < R_1$ and $\lim |a_k|^{1/k} = 1/R_2$, then $F_n(z)$ form a basis in $|z| < R_1 R_2$. The following special cases are obtained as corollaries. (A) If $f(z)$ is analytic in $|z| < R$ and $\alpha_k^{(n)}$ are constants such that, for each nonnegative integer s, $\inf_n \sum_{i=s+1}^{n} |\alpha_i^{(n)}/\alpha_s^{(n)}| = 0$, then the system $f_n(z) = \sum_{k=0}^{n} \alpha_k^{(n)} a_k z^k$ is complete in $|z| < R$. This contains older results of other authors on systems $f(\alpha_n z)$ and $f_{(n)}(z)/z^n$, where $f_{(n)}$ is the n-fold integral of $f(z)$ starting from the origin. (B) Using $g_n(z) = z^n(1-z)^{-n-1}$ it follows that $\{z^n f^{(n)}(z)\}$ is complete in any bounded simply-connected region in which $f(z)$ is analytic. (C) Using $g_n(z) = 1/(1 - \alpha_n z)$, where $\{\alpha_n\}$ has a finite limit point α, it follows that $f(\alpha_n z)$ is complete in any bounded simply-connected region in which $f(\alpha z)$ is analytic. (D) The system $f_n(z) = \sum_{k=n}^{\infty} a_k z^k$ formed from $f(z)$ is complete in any bounded simply connected region in which $f(z)$ is analytic. (E) The system $f_n(z)$ of (D) is a basis in $|z| < R$ if $f(z)$ is analytic there and $\lim |a_k|^{1/k} = 1/R$. (F) If $f(z)$ is analytic in $|z| < R$, the system $\{\sum_{k=0}^{\infty} a_{k+n} z^{kn}\}$ is complete in $|z| < R$.

R. P. Boas, Jr. (Providence, R. I.).

Combes, Jean. Sur l'uniformisation des fonctions algébroïdes. C. R. Acad. Sci. Paris 227, 1325–1326 (1948).

The author presents, without proofs, some results of his researches on the uniformisation of algebroid functions of a complex variable by means of Fuchsoid functions, which are a generalization of Fuchsian functions. For the Fuchsoid functions the fundamental field becomes a convex polygon with infinitely many sides; beside the types of vertices that appear in the Fuchsian case, vertices that are accumulation points of vertices also appear. The condition that the minimum of the distance between two nonconsecutive vertices of the polygon is greater than zero is sufficient (together with the conditions that are obtained as an immediate generalization of the Fuchsian case) in order that a polygon with infinitely many sides is the fundamental field for a Fuchsian group.

A Fuchsoid group being given, there are always functions that are automorphic for the group, and they may be obtained as quotients of two Poincaré θ-series; these functions are meromorphic in the interior of the fundamental field, invariant for the substitutions of the group, but they do not always take any value a finite number of times in the interior of the field. Finally, the author takes into consideration the Riemann surface of an integral algebroid function; on such a surface there is a linearly polymorphic function with singular points of arbitrarily fixed multiplicities at any singular point; the singularities and their multiplicities determine the linearly polymorphic function except for a projectivity. By means of such a linearly polymorphic function the uniformization of the algebroid function is realized. *F. Conforto* (Rome).

Conforto, Fabio. Alcune osservazioni sulla teoria delle funzioni e delle varietà quasi abeliane. Boll. Un. Mat. Ital. (3) 4, 6–13 (1949).

Suppose that K is a field of quasi-Abelian functions of degree of transcendency π, i.e., a field of functions with an algebraic addition theorem in the sense of Weierstrass and Painlevé. The author illustrates the fact that the following three problems do not necessarily coincide for such fields: (i) the determination of all automorphisms of K, (ii) the study of all birational transformations of the attached Picard variety, and (iii) the theory of (unit) complex multiplications of the matrix of periods in the sense of Hurwitz. It is important for these observations to remind oneself that examples of quasi-Abelian functions and varieties are furnished by generalized Jacobian varieties belonging to curves of virtual genus π and effective genus $p < \pi$, where the varieties in question are products of Abelian Picard varieties (classical theory of Riemann matrices) of dimension p and a linear space of dimension $\pi - p > 0$.

O. F. G. Schilling (Chicago, Ill.).

Toyama, Hiraku. On a non-Abelian theory of algebraic functions. Kōdai Math. Sem. Rep., no. 2, 28–35 (1949).

This is the first of a series of papers in which the author proposes to discuss the theory of non-Abelian extensions.

In this note the generalized Riemann-Roch theorem is proved for divisors of matrices on a closed Riemann surface of genus greater than unity. This theorem and the present method of proof are contained in A. Weil's paper [J. Math. Pures Appl. (9) 17, 47–87 (1938)]. *O. F. G. Schilling.*

Theory of Series

***Hardy, G. H. Divergent Series.** Oxford, at the Clarendon Press, 1949. xvi+396 pp. 30s., Great Britain; $8.00, U. S. A.

This is an inspiring textbook for students who know the theory of functions of real and complex variables and wish further knowledge of mathematical analysis. There are no problems displayed and labelled "problems," but one who follows all of the arguments and calculations of the text will find use for his ingenuity and pencil. The book deals with interesting and important problems and topics in many fields of mathematical analysis to an extent very much greater than that indicated by the titles of the chapters. It is of course an indispensable handbook for those interested in divergent series. It assembles a considerable part of the theory of divergent series which has previously existed only in periodical literature. Hardy has greatly simplified and improved many theories, theorems and proofs. In addition, numerous acknowledgments show that the book incorporates many previously unpublished results, and improvements of old results, communicated to Hardy by his colleagues and by others interested in the book.

Chapters 1 and 2 present preliminary calculations with divergent series, with remarks about contributions and views of Euler, Cauchy, Abel, Fourier, d'Alembert, Laplace and others. Chapter 3 treats the general theory of sequence-to-sequence matrix transformations, giving theorems of Toeplitz (or Silverman-Toeplitz), Kojima, Schur, W. A. Hurwitz, and related theorems. Function-to-function kernel transformations are treated more briefly in the text and a note at the end of the chapter.

Chapter 4 treats special methods for evaluation of series and sequences: Nörlund, Euler (exponential method of order 1), generalized Abel (or Dirichlet-series), Lindelöf, Mittag-Leffler, Borel (and other methods similarly defined by entire functions), Vallée Poussin, Riesz (typical), the moment constant methods, and the two different Riemann methods of order 2. Many theorems on regularity, inclusion and consistency are obtained. Chapter 5 gives a detailed treatment of the Cesàro, Hölder, and Abel methods and of the relations among them. Chapters 6 and 7 give many Tauberian theorems for Cesàro summability, and for Abel summability and generalized Abel summability. Chapter 8 treats the general Euler methods, the Borel method, and analytic continuation by these and other methods, and Chapter 9 gives Tauberian theorems for the Euler and Borel methods. Chapter 10, on multiplication of series, treats summability of Cauchy and Dirichlet products by various methods. Chapter 11 gives the theory of the transformations, introduced by W. A. Hurwitz and Silverman and by Hausdorff, which commute with the arithmetic mean transformation. Chapter 12 gives the theory of Tauberian theorems initiated by N. Wiener, with applications to number theory. Chapter 13, the final chapter, treats the Euler-Maclaurin summation formula and summability of Euler-Maclaurin series. Appendix 1 gives a number of examples of the use of divergent series for evaluation of definite integrals. Appendix 2 gives formulas needed for the study of the summability of Fourier series by various methods. Appendix 3 gives several theorems on Riemann and Abel summability. Appendix 4 relates Lambert summability and Ingham summability. Appendix 5 gives theorems of M. L. Cartwright on generalized Abel summability, with applications to Fourier series.

Each chapter is followed by notes on that chapter, and it is with reference to these notes that the reviewer must express regret. The reviewer feels that the notes bearing on the text should have been incorporated in the text or placed in footnotes so one could know of their existence when one wants to see them, that the references should be collected and alphabetized in the back of the book, and that the text should, at the appropriate places, call attention to these references. Perhaps Hardy was dissuaded from collecting his references at the end of the book by the fact, which he often acknowledges in the notes, that the list of references he gives is very far from complete. To end on a cheerful note, the reviewer is happy to report that there are, at the end, a list of 31 relevant books, a list of periodical abbreviations used, a list of 140 authors giving the sections in which they are cited, and an adequate index.

R. P. Agnew (Ithaca, N. Y.).

Ree, Rimhak. On a problem of Max A. Zorn. Bull. Amer. Math. Soc. 55, 575–576 (1949).

M. Zorn [same Bull. 53, 791–792 (1947); these Rev. 9, 139] has proved the theorem (originally proposed by the reviewer) that if every substitution $x=at$, $y=bt$ in which a and b are complex numbers transforms $\sum a_{ij}x^iy^j$ into a power series with a nonvanishing radius of convergence, then the series $\sum|a_{ij}x^iy^j|$ converges for sufficiently small $|x|$ and $|y|$, and Zorn posed the problem of whether the theorem would hold in the real number field as well. The author proves that this is indeed the case, basing his argument on the following lemma of his own. Let $P(x, y)=\sum_{i+j=n}a_{ij}x^iy^j$ be a homogeneous polynomial in real variables. If $|P(x, y)|\leqq M$ for $|x-x_0|\leqq 2\delta$, $|y-y_0|\leqq 2\epsilon$, then $|a_{ij}\delta^i\epsilon^j|\leqq M$. *S. Bochner.*

Birindelli, Carlo. Rapporti di un recente metodo di sommazione delle serie con altri e sue estensioni alle successioni e serie multiple. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. (8) 2, 37–65 (1948).

This paper generalizes the Picone class of summation methods in such a way that Picone's theorem concerning the contraction of the indetermination interval remains valid for the generalized class of methods. The contraction theorems are also studied for the analogous methods summing multiple series. *E. Kogbetliants.*

Hyslop, J. M. A theorem on the summability of series. Proc. London Math. Soc. (2) 51, 176–185 (1949).

The author considers the order of magnitude of the Rieszian means $C_\alpha(\omega)=\omega^{-\alpha}\sum_{0\leq n<\omega}(\omega-n)^\alpha a_n$ in relation to that of the means $C_{\alpha,\lambda}(\omega)$ defined for the series $\sum_0^\infty(n+k)^\lambda a_n$ by $C_{\alpha,\lambda}(\omega)=\omega^{-\alpha}\sum_{0\leq n<\omega}(\omega-n)^\alpha(n+k)^\lambda a_n$. His results are stated in the following comprehensive theorem. If $k>0$, $\alpha\geqq 0$, $\alpha-\rho>-1$, $\lambda-\rho>-1$, $\beta\geqq 0$ and if $C_\alpha(\omega)=o(\omega^{-\rho})$ then

$$\begin{aligned} C_{\beta,\lambda}(\omega) &= A+o(\omega^{\lambda-\rho})+o(\omega^{\lambda-\rho+\alpha-\beta}) && (\lambda<\rho),\\ &= o(\omega^{\lambda-\rho})+o(\omega^{\lambda-\rho+\alpha-\beta}) && (\rho<\lambda),\\ &= o(\log\omega)+o(\omega^{\alpha-\beta}) && (\lambda=\rho\neq 0, 1, 2, \cdots),\\ &= o(1)+o(\omega^{\alpha-\beta}) && (\lambda=\rho=0, 1, 2, \cdots), \end{aligned}$$

where Λ is a constant whose explicit value is given. The method makes extensive use of repeated partial integration, inversion of orders of integration and of three known lemmas on Rieszian means. *H. G. Eggleston* (Swansea).

Vernotte, Pierre. Sommation, par pondération binomiale des séries convergentes alternées qui paraissent commencer par diverger; application aux séries divergentes. C. R. Acad. Sci. Paris **228**, 1840–1842 (1949).

Without recognition that he is using the standard Euler transformation $\sigma_n=2^{-n}\sum_{k=0}^{n}\binom{n}{k}s_k$ whose theory and effectiveness are well known, the author claims it to be a "résultat nouveaux et curieux" that σ_n is sometimes a much better approximation to the value of a series $\sum u_n$ than $s_n=u_0+\cdots+u_n$ is. *R. P. Agnew* (Ithaca, N. Y.).

Pini, Bruno. Convergenza, fattori di convergenza, convergenza generalizzata per determinanti infiniti. Rend. Sem. Mat. Univ. Padova **17**, 160–185 (1948).

The author applies the notion of convergence factors to the evaluation of determinants of infinite matrices. Given a matrix A, diagonal matrices $D_1(x)$ and $D_2(x)$ are chosen, with entries functions. The generalized determinant of $I+A$ is defined to be $\lim_{x\to x_0}\det(I+D_1AD_2)$. The author obtains conditions under which this is regular.

R. C. Buck (Madison, Wis.).

Korenblyum, B. I. On theorems of Tauberian type. Doklady Akad. Nauk SSSR (N.S.) **64**, 449–452 (1949). (Russian)

The author applies previous results of his to the derivation of generalizations of Tauberian theorems of Wiener, Bochner and Hardy, Hardy and Littlewood, Szász, etc. These earlier results [same vol., 281–284 (1949); these Rev. **10**, 462] are specialized by taking the group in question to be the reals under addition and by setting $\theta(x)=e^x$ for $x<0$ and $\theta(x)=0$ for $x\geqq 0$. A typical theorem, valid for measurable functions f on $(0,\infty)$ such that

$$\sup_{0<h<\infty} h^{-1}\int_0^h |f(x)|^p dx<\infty,$$

runs as follows in the case $p=1$. If $f(x)$ is $(C,1)$ limitable to A as $x\to 0$ (or ∞, respectively), then for any continuous function k on $(0,\infty)$ such that

$$\int_0^\infty \left\{\sup_{x\leq t<\infty}|k(t)|\right\}dx<\infty,$$

$$\lim_{\epsilon\to 0(\infty)}\int_0^\infty k(\epsilon x)f(x)dx=A\int_0^\infty k(x)dx;$$

and if $f(x)\geqq 0$, this remains true when k has a finite number of discontinuities of the first kind. No proofs are given.

I. E. Segal (Chicago, Ill.).

Popović, Božidar. Sur un théorème relatif aux valeurs asymptotiques de l'intégrale de Laplace-Abel. Acad. Serbe. Bull. Acad. Sci. Mat. Nat. A. **7**, 5–11 (1941).

This is concerned with Tauberian conditions on $A(t)$ in

$$J(\sigma)=\sigma\int_0^\infty e^{-\sigma t}A(t)dt, \qquad \sigma>0,$$

sufficient to secure the validity of the inference from $J(\sigma)\sim\varphi(1/\sigma)$ to $\Gamma(s+1)A(x)\sim\varphi(x)$ $(\sigma\to 0;\ x\to\infty)$, where $\varphi(x)=x^sL(x)$ $(s>-1)$ is a function of regular growth. A typical (known) sufficient condition is

$$(1)\qquad \liminf\{A(t')-A(t)\}/\varphi(t)\geqq -w(\lambda)$$

$(t\leqq t'\leqq\lambda t,\ t\to\infty)$, where $w(\lambda)\to 0$ as $\lambda\downarrow 1$. It is shown here that we may remove $\varphi(t)$ from the left hand side of (1), and transfer its effect to the length of interval over which t' may range by changing λt to $T=V\{\lambda\Lambda(t)\}$, where V is the function inverse to $\Lambda(t)=\exp\int_1^t u^{-1}\varphi(u)du$, provided that $\varphi(x)\geqq\delta>0$ $(x\geqq 0)$. This remark is developed in various ways, and an example is given to show that, in one instance at any rate, the result is nearly the best possible of its kind. Comparison with other work is not facilitated by the numbered references to the literature, since no bibliography is visible. *A. E. Ingham* (Cambridge, England).

Fourier Series and Generalizations, Integral Transforms

Bari, N. K. The uniqueness problem of the representation of functions by trigonometric series. Uspehi Matem. Nauk (N.S.) **4**, no. 3(31), 3–68 (1949). (Russian)

An exhaustive review, in many cases accompanied by complete proofs, of the existing results in the theory of uniqueness of representation of functions by trigonometric series. *A. Zygmund* (Chicago, Ill.).

Men'šov, D. On the Fourier series of continuous and summable functions. Doklady Akad. Nauk SSSR (N.S.) **67**, 787–789 (1949). (Russian)

A previous result of the author [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 245–246 (1941); Rec. Math. [Mat. Sbornik] N.S. **11(53)**, 67–96 (1942); these Rev. **3**, 106; **7**, 59] asserts that given any continuous function $f(x)$ of period 2π and any number $\epsilon>0$, we can change the function f in a set E of measure less than ϵ in such a way that the Fourier series of the new function converges uniformly. This set E depends, in general, not only on ϵ but also on f. It is now stated that, if the modulus of continuity $\omega(\delta)$ of f does not exceed a nondecreasing function $\rho(\delta)$, tending to 0 with δ, then the set E may be selected independently of f. It will then depend on ϵ and $\rho(\delta)$ (the latter function is fixed, but it may tend to zero arbitrarily slowly). Let P be any perfect nondense set in $(0,2\pi)$. The author quotes as an unsolved problem whether any continuous and periodic function $f(x)$ can be so changed outside P that the resulting continuous function $\psi(x)$ has an almost everywhere convergent Fourier series. The problem remains open even if we only require that ψ be quadratically integrable. On the other hand, the following theorem is stated with some hints as to the proof. For any $f\in L$ and any perfect nondense set $P\subset(0,2\pi)$, we can change f outside P so that the resulting function $\psi(x)$ is L-integrable and its Fourier series converges almost everywhere. *A. Zygmund.*

Gonzalez Baz, Enriqueta. Theorems on the absolute convergence of Fourier series. Comisión Impulsora y Coordinadora de la Investigación Científica (Mexico). Anuario **1945**, 43–49 (1947). (Spanish)

Known proofs of known theorems. *R. C. Buck.*

Zamansky, Marc. Sur la série de Fourier et la série conjuguée d'une fonction continue périodique. C. R. Acad. Sci. Paris **228**, 1687–1689 (1949).

Remarks on the degree of approximation of f and $\tilde{f}$ by the partial sums of the Fourier series of f and of the conjugate series. *A. Zygmund* (Chicago, Ill.).

Zamansky, Marc. Sur l'approximation des fonctions continues périodiques. C. R. Acad. Sci. Paris **228**, 1838–1840 (1949).

Let $g(u)$ be a polynomial in u satisfying the conditions $g(0)=1, g(1)=0, g'(0)=\cdots=g^{(p-1)}(0)=0, g^{(p)}(0)\neq 0$. Given any continuous $f(x)\sim a_0/2+\sum(a_n\cos nx+b_n\sin nx)$, the author considers the approximation of $f(x)$ by the trigonometric polynomials

$$T_n^p(x)=\tfrac{1}{2}a_0+\sum_{k=1}^{n}g(k/n)(a_k\cos kx+b_k\sin kx).$$

The approximation of f by $T_n^p(x)$ cannot be better than $O(1/n^p)$. This approximation is achieved for p odd if $\tilde{f}^{(p-1)}\in\mathrm{Lip}\,1$, and for p even if $f^{(p-1)}\in\mathrm{Lip}\,1$.

A. Zygmund (Chicago, Ill.).

Zamansky, M. Classes de saturation de certains procédés d'approximation des séries de Fourier des fonctions continues et applications à quelques problèmes d'approximation. Ann. Sci. École. Norm. Sup. (3) **66**, 19–93 (1949).

The paper contains complete proofs of results announced by the author in several other notes [C.R. Acad. Sci. Paris **224**, 704–706 (1947); **226**, 1066–1068 (1948); **227**, 463–464, 1011–1013 (1948); **228**, 460–461 (1949); these Rev. **8**, 457; **9**, 582; **10**, 113, 292, 449; and the two notes reviewed above.]

A. Zygmund (Chicago, Ill.).

Boas, Ralph P. Sur les séries et intégrales de Fourier à coefficients positifs. C. R. Acad. Sci. Paris **228**, 1837–1838 (1949).

Two generalizations are given of a result of Dugué [same vol., 1469–1470 (1949); these Rev. **10**, 516] concerning the analytic character of a function having a Fourier cosine series with positive coefficients. (1) If $f(x)=\int_{-\infty}^{\infty}e^{ixt}d\alpha(t)$ where $\alpha(t)$ is bounded and nondecreasing and if $f(x)$ coincides with an analytic function in a real neighborhood of $x=0$, then the same is true for all real x. Furthermore, there is an analytic extension of $f(x)$ in some band $\Im(z)<\delta$ of the $z\,(=x+iy)$ plane. (2) If a function analytic in the interval $-\delta\leqq\theta\leqq\delta$ has a Fourier series $\sum_{-\infty}^{\infty}c_ne^{in\theta}$ with positive coefficients, and if the sequence of values of n at which c_n changes sign possesses a nonzero asymptotic density $1/R$, then if $R>\pi/\delta$ the function is analytic for all real θ.

P. Civin (Eugene, Ore.).

Redheffer, R. M. Design of a circuit to approximate a prescribed amplitude and phase. J. Math. Physics **28**, 140–147 (1949).

The problem of approximating a desired complex valued function of frequency ω by the transfer function of a minimum-phase network becomes, after setting $\omega=\tan(\phi/2)$, the problem of approximating a desired function $a(\phi)+ib(\phi)$ in $(0,\pi)$ by a Fourier polynomial $\sum_0^n a_ke^{ik\phi}$ with real coefficients. If the a_k are chosen to minimize the mean square error over $(0,\pi)$, the error will generally not tend to zero as n increases. If they are chosen to minimize the mean square error of the approximation in a subinterval $(0,s)$, this error tends to zero. The system of linear equations to be solved for the a_k is given as well as the system for the coefficients of the best mean square approximation constrained to pass through some fixed points.

E. N. Gilbert.

Wang, Fu Traing. On Riesz summability of Fourier series. III. Proc. London Math. Soc. (2) **51**, 215–231 (1949).

[For parts I and II see same Proc. (2) **47**, 308–325 (1942); J. London Math. Soc. **17**, 98–107 (1942); these Rev. **4**, 37.]

The present paper is devoted to the study of Riesz summability $(R, \exp(\log w)^\beta, k)$ of the Fourier series of a periodic integrable function $f(x)$ for $1<\beta$ and $k>0$. If the Fourier coefficients of $f(x)$ are denoted by a_n and b_n, then, for $1<\beta<2$, almost everywhere summability is obtained for any $k>0$ if $\sum_{n=2}^{\infty}(a_n^2+b_n^2)(\log n)^{\beta-1}$ is convergent. A pointwise summability theorem is obtained as well as a one-sided order condition on $(a_n\cos nx+b_n\sin nx)$ which assures the convergence of the Fourier series.

P. Civin.

Mitra, S. C. On a method of summing divergent series. J. Indian Math. Soc. (N.S.) **13**, 4–16 (1949).

The Bessel (J, r) sum of a series with partial sums s_n is defined for all real r as the limit (if such exist) of $2\sum_{k=0}^{\infty}J_{r+k}^2(w)s_k$ as $w\to\infty$. The method is regular [Cooke, J. London Math. Soc. **12**, 299–304 (1937)]. The author shows summability $(J, 0)$ of Legendre and $(J, \frac{1}{2})$ of Bessel series of fairly regular functions at certain exceptional points where the hypotheses do not insure convergence. The summability $(J, 0)$ of the conjugate Fourier series and $(J, \frac{1}{2})$ of the derived Fourier series are demonstrated at points satisfying some auxiliary conditions. [The definition stated in the paper for summability (J, r) is faulty and the one actually used is that quoted above.]

P. Civin.

Čakalov, L. N. On the convergence of a formula of trigonometric interpolation. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 177–191 (1949). (Russian)

Suppose $f(x)$ is defined for all x and is bounded and periodic of period 2π. The author begins with a proof that the Jackson polynomials

$$Q_n(x,f)=n^{-2}\sum_{k=0}^{n-1}\frac{1-\cos nx}{1-\cos(x-x_k)}f(x_k),$$

where $x_k=2k\pi/n$, converge to $f(x)$ at all points of continuity of $f(x)$. He then goes on to his main investigation, which is to discover the behaviour of the Jackson polynomials at a point a of simple discontinuity of $f(x)$. Here he finds that the behaviour depends on the arithmetic character of the point a. If $a/2\pi$ is irrational then the limiting values of $Q_n(a, f)$ is the nondenumerable set of all numbers λ lying between $f(a-0)$ and $f(a+0)$. If $a/2\pi$ is a rational number p/q, say, where $q>1$, then $Q_n(a, f)$ has the q limiting values $f(a)$ and

$$\tfrac{1}{2}\{f(a+0)+f(a-0)\}+\{f(a+0)-f(a-0)\}\psi((2r-q)/q),$$

where

$$\psi(x)=\int_0^1(1-t)\sin\pi xt\csc\pi t\,dt$$

and $r=1, 2, \cdots, q-1$.

A. C. Offord (London).

Orlicz, W. Sur quelques propriétés des fonctions de Baire périodiques. Studia Math. **10**, 148–158 (1948).

The author proves again a theorem of his that for any measurable periodic function $f(x)$ the relation

$$\limsup_{n\to\infty}f(\omega_nx+\theta_n)=\sup{}_L f(x)$$

holds almost everywhere ($\sup_L$ denotes the essential least upper bound) and he shows that for Baire functions the assertion remains in force if every "exceptional set" which occurs is interpreted to mean a set of the first category. Also if a family $\{f_n(x)\}$ is suitably compact, then the rela-

tion can be generalized to

$$\limsup_{n\to\infty} f_n(\omega_n x+\theta_n)=\limsup_{n\to\infty}\sup_L f_n(x)$$

both for measurable and for Baire functions.

S. Bochner (Princeton, N. J.).

Hartman, S., Steinhaus, H., et Fast, H. Sur les presque-périodes des fonctions périodiques. Colloquium Math. 1, 297–304 (1948).

Let $\delta_f(\epsilon)$ denote the supremum of the $\delta>0$ for which $|x'-x''|<\delta$ implies $|f(x')-f(x'')|<\epsilon$ for a given continuous periodic function $f(x)$, $-\infty<x<\infty$, and call a number $\tau_f(\epsilon)$ an ϵ-period if $|f(x+\tau_f(\epsilon))-f(x)|<\epsilon$ for all x. A $\tau_f(\epsilon)$ is called nonsingular if its distance from the nearest period is less than $\delta_f(\epsilon)$. Call f nonsingular if there exists an $\epsilon_0>0$ such that for $\epsilon<\epsilon_0$ every $\tau_f(\epsilon)$ is nonsingular. An example of a singular function is constructed and a sufficient condition for a function to be nonsingular is given. A theorem of Gottschalk [Bull. Amer. Math. Soc. 52, 633–636 (1946); these Rev. 8, 34] is proved. *E. Følner* (Copenhagen).

Rodriguez Vidal, R. Double asymptotically quasiperiodic sequences. Revista Mat. Hisp.-Amer. (4) 8, 239–242 (1948). (Spanish)

A double sequence a_m^n; $m, n=0, \pm1, \pm2, \cdots$, is called almost periodic when there corresponds to every $\epsilon>0$ a natural number L such that every square with side L contains a point (l_1, l_2) with integral coordinates satisfying the inequality $|a_{m+l_1}^{n+l_2}-a_m^n|<\epsilon$ for all values of m and n. [The author considers only squares with center on the diagonal $l_2=l_1$, but his applications show that this is a mistake]. A sequence b_m^n is called asymptotically almost periodic if $b_m^n=a_m^n+\epsilon_m^n$, where a_m^n is almost periodic, while $\epsilon_m^n\to0$ when $m^2+n^2\to\infty$. This decomposition is unique. The author introduces a Fourier series corresponding to b_m^n. The "principal" part a_m^n is uniquely determined by this Fourier series.

H. Tornehave (Copenhagen).

Cherry, T. M. On expansions in eigenfunctions, particularly in Bessel functions. Proc. London Math. Soc. (2) 51, 14–45 (1949).

The author establishes the integral theorems of Fourier, Hankel and Weber in forms valid for functions exponentially large at infinity. The method is essentially that used by Titchmarsh [Eigenfunction Expansions Associated with Second-Order Differential Equations, Oxford, 1946; these Rev. 8, 458]. *H. Pollard* (Ithaca, N. Y.).

Agranovič, Z. On some questions connected with equations of Sturm-Liouville type on a semi-axis. Doklady Akad. Nauk SSSR (N.S.) 66, 1025–1028 (1949). (Russian)

Harmonic analysis of even functions on the reals is generalized by replacing the functions $\cos\lambda^{\frac{1}{2}}x$ which play a basic role there by the functions $\varphi(x,\lambda)$ which satisfy the equations $\varphi''-(\rho(x)-\lambda)\varphi=0$, $\varphi(0,\lambda)=1$, $\varphi_x'(0,\lambda)=0$, where ρ is an even continuous function. This continues work of A. Povzner [Mat. Sbornik N.S. 23(65), 3–52 (1948); these Rev. 10, 299]. Many theorems are stated, mostly with no indications of their proofs, among which are partial analogs of Wiener's Tauberian theorem and of a theorem of Beurling concerning approximation by linear combinations of translates of functions. With T_x^y defined as in Povzner's paper, let B be the Banach space of measurable functions $f(x)$ on $(0, \infty)$ defined as follows: (a) when

$$\sup_{x,y}|T_x^y(f)|\leqq K\sup_x|f(x)|$$

for some constant K, $B=L(0, \infty)$; (b) otherwise, B consists of the functions f for which $\int_0^\infty(1+x)|f(x)|dx<\infty$, and has the corresponding norm. For $f\in B$, the function of λ $(0\leqq\lambda<\infty)$ $\Phi_f(\lambda)=\int_0^\infty f(x)\varphi(x,\lambda)dx$ is called the S-L transform of f, and the set of values of λ for which $\Phi_f(\lambda)\not\equiv0$ is called the S-L spectrum of f. Typical results are as follows. If $f\in B$ and if every λ $(\geqq0)$ is in the S-L spectrum of f, then every element of B is a limit of a sequence of finite linear combinations of the (generalized translates) $T_x^t(f)$, $-\infty<t<\infty$. If λ is in the S-L spectrum of the bounded continuous function f on the reals, and if case (a) above holds, then $\varphi(x,\lambda)$ is the limit in the "p-metric" of a sequence of finite linear combinations of generalized translates of f; here the p-norm of a function g is $\sup_x|g(x)p(x)^{-1}|$ and p is restricted to be in the class of continuous even functions p on the reals such that $p(x)>0$ for all x, $p(x)$ is nondecreasing for $x>0$, and $p(x)\to\infty$ as $x\to\infty$ (analogous results for case (b)). *I. E. Segal* (Chicago, Ill.).

Prachar, K. Zur Eulerschen Summierung Neumannscher und Legendrescher Reihen. Monatsh. Math. 53, 138–150 (1949).

The author proves that the ellipse of convergence of the Neumann expansion of a function $f(z)$ of the complex variable z in a series of Legendre polynomials $P_n(z)$, namely (I) $f(z)\sim\sum_{n=0}^\infty a_nP_n(z)$ with $2\pi ia_n=(2n+1)\int cf(t)Q_n(t)dt$, belongs to the interior of a larger domain D_E, in which the expansion (I) is summable by Euler's method of first order. This result is a generalization of Knopp's theorem on the summability of the Maclaurin expansion by Euler's method. The shape of D_E is determined by the singularities of $f(z)$ and the author describes how to construct this domain for a given $f(z)$. In the last two sections the author shows that the Euler-Lebesgue constants of the Legendre series at the point $x=1$ increase as $n^{\frac{1}{2}}$, when n increases, so that there are continuous functions $f(x)$ whose Legendre series are not summable by Euler's method at the point $x=1$.

E. Kogbetliantz (New York, N. Y.).

Bose, S. K. A study of the generalised Laplace integral. I. Bull. Calcutta Math. Soc. 41, 9–27 (1949).

The integral in question is (1): $\varphi(p)=p\int_0^\infty K(px)h(x)dx$, where (2): $K(z)=(2z)^{-\frac{1}{2}}W_{k,m}(2z)$ and W is Whittaker's confluent hypergeometric function. This "Whittaker transform" was introduced by R. S. Varma [Current Sci. 16, 17–18 (1945); these Rev. 9, 346]. In section I of the present paper the author gives a considerable number of transform-pairs. In section II he derives a number of rules which correspond to known rules of the Heaviside calculus. [Actually, these rules hold for all the transforms of the form (1) with a few qualitative restrictions on K, and with the exception of the evaluation of one integral do not depend at all on the particular form (2).] A few examples for the application of these rules follow. Section III is devoted to a general theorem giving the connection between $\varphi(p)$ and the function whose operational image (in the Heaviside calculus) is $h(p)p^{1-\alpha}(1+p)^{-\beta}$, and to several examples of this theorem. *A. Erdélyi* (Pasadena, Calif.).

Polynomials, Polynomial Approximations

***Guillet, A., Aubert, M., et Parodi, M. Propriétés des polynomes électrosphériques.** Mémor. Sci. Math., no. 107. Gauthier-Villars, Paris, 1948. iv+58 pp.

The following problem of physics: given two nonintersecting, electrified spheres at distinct and known potentials, find the distributions of densities, the capacities and the coefficient of mutual induction, etc., can be solved with the aid of series involving the polynomials $U_n(x)$, where $U_n(\cos\alpha)=\sin[(n+1)\alpha]/\sin\alpha$. The authors give to these (and related) polynomials the name of electrospherical and study their elementary properties in all details.

E. Kogbetliantz (New York, N. Y.).

Hahn, Wolfgang. Über Orthogonalpolynome, die q-Differenzengleichungen genügen. Math. Nachr. **2**, 4–34 (1949).

Let M_n $(n\geqq 0)$ be a sequence for which the determinants $D_{0,n}=|M_{i+k}|$ $(0\leqq i,k\leqq n-1)$ are all nonzero, and let

$$(1)\qquad p_n(x)=D_{0,n}^{-1}\cdot\begin{vmatrix} M_0 & \cdots & M_n \\ \cdots & & \cdots \\ M_{n-1} & \cdots & M_{2n-1} \\ 1 & \cdots & x^n \end{vmatrix},$$

so that the polynomials $p_n(x)$ satisfy a recurrence formula of the type (2) $p_n(x)=(a_nx+b_n)p_{n-1}(x)+c_np_{n-2}(x)$. If all the $D_{0,n}$ are positive, it is known that there exist an interval (a,b) and a nondecreasing weight-function $\psi(x)$ such that $\int_a^b x^n d\psi(x)=M_n$, $\int_a^b p_n(x)p_m(x)d\psi(x)=0$ for $m\neq n$. Let $Lf(x)=(f(qx+\omega)-f(x))/((q-1)x+\omega)$, $|q|\neq 1$. Then the following properties all define the same class of polynomials. (A) $\{Lp_n(x)\}$ is an orthogonal system along with $\{p_n(x)\}$. (B) The $p_n(x)$ satisfy

$$(a_{11}x^2+a_{12}x+a_{13})L^2y+(a_{21}x+a_{22})Ly+a_3y=0,$$

the coefficients being independent of n. (C) There is a generalized Rodrigues formula:

$$p_n(x)=(\pi(x))^{-1}L^n(f_0(x)f_1(x)\cdots f_n(x)\pi(x)),$$

$f_i(x)=f_{i+1}(qx+\omega)$, with $\pi(x)$, $f_0(x)$ independent of n. (D) If $p_n(x)=\sum_{i=0}^n a_{n,i}x^i$, there exist polynomials $g_1(u,v)$, $g_2(u,v)$ for which $g_1([n],[i])a_{n,i}=g_2([n],[i])a_{n,i-1}$, the arguments being the basic numbers $[r]=(q^r-1)/(q-1)$. (E) The moments M_n are given by

$$M_n=M_{n-1}(A+Bq^n)/(C+Dq^n),\quad AD-BC\neq 0,\quad 0<q<1.$$

The general solution of the functional equation in (B) is obtained in terms of the Heinean series ${}_3\varphi_2$, omitting the troublesome cases in which the parameter-differences are integral, when logarithms must be introduced to obtain a second solution. It is shown that (A) implies (B), even when orthogonality is replaced by (2) above, and the $p_n(x)$ are not assumed to be polynomials. The general polynomial solutions are exhibited in the form ${}_3\varphi_2(q^{\alpha+n},q^{-n},x;q^\beta,q^\gamma;q)$. It is then proved that (C) implies (B), and a simple functional equation for $\pi(x)$ is derived. If the coefficients in (B) are known, $\pi(x)$, $\psi(x)$, and the interval of orthogonality can be determined by means of explicit formulas. The proof that (D) implies (A) is followed by a complete discussion of the moment-problem (E). After evaluating the Hankel determinant $D_{r,s}=|M_{j+k}|$ $(r\leqq j<r+s;\ 0\leqq k<s)$, the author shows that the only solvable cases are (i) $AB<0$, (ii) $B=0$, $CD>0$, and that the problem is undetermined if and only if $C=0$. He gives explicit solutions to which all solvable cases may be reduced, and a second solution for the undetermined case. The paper ends with a discussion of six general types of q-polynomials, which, together with their limiting cases for $q\to 1$ (Jacobi, Laguerre, Hermite, Charlier, Bateman polynomials, and others) are related to the existing literature with plentiful references. [The paper contains a number of minor typographical errors.] *N. J. Fine.*

Reuter, G. E. H. On the boundedness of the Hermite orthogonal system. J. London Math. Soc. **24**, 159–160 (1949).

Let $\phi_n(x)=\pi^{-\frac{1}{4}}(2^n n!)^{-\frac{1}{2}}e^{\frac{1}{2}x^2}(d/dx)^n e^{-x^2}$. We have

$$\int_{-\infty}^{\infty}\phi_n^2(x)dx=1.$$

The author proves that, for all n and all real x,

$$|\phi_n(x)|\leqq 2^{\frac{1}{2}}\phi_0(0)=(2/\pi)^{\frac{1}{4}},$$

and conjectures that the factor $2^{\frac{1}{2}}$ can be replaced by 1.

G. Szegő (Stanford University, Calif.).

Hirschman, I. I., Jr., and Widder, D. V. Generalized Bernstein polynomials. Duke Math. J. **16**, 433–438 (1949).

A new proof of the theorem of Müntz on the approximation of a continuous function $f(t)$, $0\leqq t\leqq 1$ by linear combinations of powers t^{α_k}, where the approximating polynomial is given by a simple explicit formula. Making the change of variables $t=e^x$ the authors prove that

$$P_n(x)=\sum_{k=0}^n f(-\sigma_{nk})H_{nk}(x)$$

is uniformly convergent to the continuous function $f(x)$, $-\infty\leqq x\leqq 0$, if $0=a_0<a_1<\cdots<a_n\to+\infty$, $\sum a_k^{-1}=+\infty$, $\sigma_{nk}=a_k^{-1}+\cdots+a_n^{-1}$ and $H_{nk}(x)$ is the solution of the differential equation

$$(1-a_{k+1}^{-1}D)\cdots(1-a_n^{-1}D)y=e^{a_kx}$$

which vanishes at $x=0$ with its first $n-k-1$ derivatives. Thus H_{nk} is a linear combination of $e^{a_kx},\cdots,e^{a_nx}$. In the case $a_k=k$, $P_n(\log t)$ is only slightly different from the classical Bernstein polynomial. *G. G. Lorentz.*

Vorob'ev, Yu. V., and Drozdovič, V. N. On methods for investigating the stability of systems of regulation with distributed parameters. Avtomatika i Telemehanika **10**, 77–83 (1949). (Russian)

The authors discuss the number of roots with positive real part of the characteristic equation

$$\Delta(\lambda)=R(\lambda)e^{\lambda}+Q(\lambda)e^{-\lambda}+P(\lambda)=0$$

with P, Q, R polynomials. If the coefficients of the polynomials depend on physical parameters e_1, e_2, the equations $\Re\Delta(i\omega)=\Im\Delta(i\omega)=0$ define a set of curves separating the (e_1,e_2)-plane into regions within which the number of roots λ with positive real part is the same. The derivatives of $\Re\Delta(i\omega)$ with respect to e_1 and e_2 along these curves determines whether the number of roots with positive part increases or decreases as the curve is crossed in a given direction. Then a region of stability can be detected if the characteristic equation can be solved for a single pair of values e_1, e_2. For given numerical values of the coefficients, a paremeter τ is introduced into the characteristic equation by replacing e^{λ} and $e^{-\lambda}$ by $e^{\lambda\tau}$ and $e^{-\lambda\tau}$. The values of τ for which the equation has a pure imaginary root can be obtained by finding the roots of a certain polynomial. Thus the stability of the desired system $(\tau=1)$ is related to the roots of the polynomial obtained for $\tau=0$.

E. N. Gilbert (Murray Hill, N. J.).

Love, E. R. On the zeros of partial sums of a certain power series. J. London Math. Soc. 24, 112–120 (1949).

Let m and n be positive integers; μ and r, real numbers with $-1<\mu<1$ and $0<r<1$; $P_n(\mu)$, the Legendre polynomial of order n; and $F_m(z)=\sum_{n=0}^{m-1}P_n(\mu)z^n$. By establishing some inequalities derived from the Laplace integral for $P_n(\mu)$, the author is able to prove that $F_m(z)\neq 0$ in $|z|\leqq r$ whenever r satisfies the inequality

$$r^{m+1}\left\{(1/2\pi)(1+3\sin\alpha)^{\frac{1}{2}}\int_0^{\pi}(1-\sin^2\alpha\sin^2\phi)^{m/2}d\phi\right.$$
$$\left.+[r^{-2}\sin\alpha+(1-\sin\alpha)\max(\tfrac{1}{4},r^{-2}-1)]^{\frac{1}{2}}|P_m(\cos\alpha)|\right\}$$
$$<(1-r^2)^{\frac{1}{2}}.$$

As a corollary, he deduces that $F_m(z)\neq 0$ in $|z|\leqq\frac{1}{2}$.

M. Marden (Milwaukee, Wis.).

Special Functions

Colombo, Giuseppe. Sulle frequenze e sullo smorzamento delle oscillazioni di una sfera vibrante radialmente in un fluido. Rend. Sem. Mat. Univ. Padova 17, 107–114 (1948).

The author proves that the roots of the equation

$$\frac{(1+px^2)\tanh x-x}{x-\tanh x}=-\frac{mx^2}{1+nx}$$

have negative real parts, and investigates in particular the behaviour of the roots for large and small values of pn/m. This enables him to investigate the characteristic frequencies, and the accompanying attenuation, of the radial oscillations of a sphere which is immersed in a fluid.

A. Erdélyi (Pasadena, Calif.).

Bancroft, T. A. Some recurrence formulae in the incomplete beta function ratio. Ann. Math. Statistics 20, 451–455 (1949).

The incomplete beta function can be expressed in terms of Gauss's hypergeometric series; and the well-known relations between contiguous functions give recurrence formulae for the incomplete beta function. These formulae can be used for the numerical computation of the incomplete beta function outside the range of Pearson's tables. An extension of Pearson's tables is now under preparation by the author.

A. Erdélyi (Pasadena, Calif.).

von Bohl, Johann Georg. Zur Lösung eines von Langmuir behandelten Integrals. Arch. Math. 1, 402–407 (1949).

Expression of

$$\int_0^{\Phi}[\Phi^{\frac{1}{2}}+\alpha\{(1-\Phi)^{\frac{1}{2}}-1\}]^{-1}d\Phi$$

in terms of elliptic integrals of the first and second kind.

A. Erdélyi (Pasadena, Calif.).

Meixner, Josef. Reihenentwicklungen vom Siegerschen Typus für die Sphäroid-Funktionen. Arch. Math. 1, 432–440 (1949).

Spheroidal wave functions are solutions of the differential equation

$$(1)\qquad (\xi^2-1)\frac{d^2S}{d\xi^2}+2\xi\frac{dS}{d\xi}+\left\{\gamma^2\xi^2-\frac{\mu^2}{\xi^2-1}-\lambda\right\}S=0.$$

The author considers the case in which λ is one of the characteristic values for which the differential equation has a solution bounded in the closed interval $(-1, 1)$. In this case he obtains expansions of any solution of (1) in infinite series of products of Bessel functions of variables $\frac{1}{2}\gamma\{\xi\pm(\xi^2-1)^{\frac{1}{2}}\}$. The expansions are analogous to Sieger's expansions of Mathieu functions. For the coefficients of the present expansions, four-term recurrence relations are obtained. If λ is known, the coefficients may be computed from these recurrence formulas. However, in this process, the loss of accuracy is considerable, and the author suggests an alternative method, based on a connection which he establishes between the coefficients of the present expansion and the coefficients of the expansion of the same spheroidal function in infinite series of Legendre functions. For the latter coefficients, numerical tables are already available.

A. Erdélyi (Pasadena, Calif.).

Koschmieder, Lothar. Die Krümmung des Schaubildes der Jacobischen Thetafunktion. Revista Mat. Hisp.-Amer. (4) 8, 257–260 (1948).

Wintner has proved that the graph of

$$y=\theta(x)=1+2\sum_{n=1}^{\infty}q^{n^2}\cos nx$$

$(0<q<1)$ has just one pair of inflection points over the period [Duke Math. J. 15, 407–411 (1948); these Rev. 10, 115]. The author examines

$$\theta(2\pi v+\pi)=\theta_0(v)=1+2\sum_{n=1}^{\infty}(-1)^nq^{n^2}\cos 2\pi nv$$

and shows that the graph of $\theta_0(v)$ has just one inflection point in the interior of $0\leqq v\leqq\frac{1}{2}$, and consequently one pair in $0\leqq v\leqq 1$. In Jacobi notation:

$$\tfrac{1}{2}K^{-1}\theta_0'(v)/\theta_0(v)=Z(u)=E(u)-(E/K)u\quad(u=2Kv),$$
$$Z'(u)=\mathrm{dn}^2u-E/K;\quad Z''(u)=-2k^2\,\mathrm{sn}\,u\,\mathrm{cn}\,u\,\mathrm{dn}\,u<0$$

$(0<u<K)$. In $(0, K)$ $Z'(u)$ decreases steadily from $1-E/K>0$ to $k'^2-E/K<0$ and so there is just one value u_0 $(0<u_0<K)$ with $Z'(u_0)=0$. Then

$$Z'(u)>0\quad(0<u<u_0);\qquad Z'(u)<0\quad(u_0<u<K);$$
$$Z(0)=0;\qquad Z(u)>0\quad(0<u<u_0).$$

In (u_0, K), $Z(u)$ steadily decreases from $Z(u_0)>0$ to $Z(k)=0$. The function $\frac{1}{2}K^{-2}\theta_0''(v)/\theta_0(v)=Z'(u)+\{Z(u)\}^2>0$ in $0<u<u_0$, and this function decreases steadily from $\{Z(u_0)\}^2>0$ to $k'^2-E/K<0$ in $u_0<u<K$. Consequently $\theta_0(v)$ has just one point of inflection in $(0, \frac{1}{2})$ and $\theta(x)$ has just two points of inflection over the period.

S. C. van Veen (Delft).

Wintner, Aurel. On a decomposition into singularities of theta-functions of fractional index. Amer. J. Math. 71, 105–108 (1949).

The object of this note is to determine the singularities of

$$(1)\qquad F(x)=1+2\sum_{n=1}^{\infty}q^{n^\lambda}\cos 2\pi nx,$$
$$0<\lambda<1,\ 0<q<1.$$

For real x, $F(x)$ is even, of period 1, and possesses derivatives of arbitrarily high order. But $F(x)$ cannot be analytic at every real x, since the coefficients of (1) do not tend to 0 as fast as the terms of a convergent geometric progression. The result is based on an expansion derived by the author in a former paper [Duke Math. J. 8, 678–681 (1941); these

Rev. 3, 232]:

$$(2)\qquad f(x)=\int_0^\infty e^{-\lambda}\cos xt\,dt=\sum_{m=0}^{\infty}c_m/x^{(m+1)\lambda+1},\quad 0\leqq x<\infty,$$

where $c_m=(-1)^m\Gamma\{(m+1)\lambda\}\sin\{\frac{1}{2}\pi(m+1)\lambda\}\Gamma^{-1}(m+1)$. By Poisson's summation formula the author finds:

$$\sum_{n=-\infty}^{\infty}f(px+pn)=2p^{-1}\int_0^\infty f(t)\cos(2\pi nt/p)\,dt$$

$$=2\pi p^{-1}\left(1+2\sum_{n=1}^{\infty}e^{-(2\pi/p)\lambda}\cos 2\pi nx\right).$$

By (2) he obtains the expansion

$$(3)\quad 1+2\sum_{n=1}^{\infty}e^{-(2\pi/p)\lambda}\cos 2\pi nx=(2\pi)^{-1}\sum_{m=0}^{\infty}c_m p^{-(m+1)\lambda}$$

$$\times\left\{\sum_{n=0}^{\infty}(n+x)^{-(m+1)\lambda-1}+\sum_{n=1}^{\infty}(n-x)^{-(m+1)\lambda-1}\right\},\quad p>0.$$

Here x and $1-x$, hence $n+x$ and $n-x$, are positive and all exponents refer to the real-valued (i.e., positive) determination of the respective multi-valued functions. The series is of the Mittag-Leffler type, but consists of branch points, instead of poles. The above results (2) and (3) differ from those given by the author. Instead of (2) he gives $\sum_{m=0}^{\infty}c_m/x^{m+1+\lambda}$, while the correct result is given in the cited article.] *S. C. van Veen.*

Differential Equations

Ghurye, S. G. On the arbitrary constants in the solution of simultaneous linear differential equations. J. Univ. Bombay (N.S.) 17, part 5, sect. A, 5–8 (1949).

For a system of two linear differential equations with constant coefficients, relations between the constants in the complementary functions are discussed. Correct ones are obtained for cases where the relations involve the particular integrals used. *P. Franklin* (Cambridge, Mass.).

Karapandjitch, G. Sur l'intégration d'une équation différentielle ordinaire. Bull. Soc. Math. Phys. Serbie 1, 29–30 (1949). (Serbian. Russian and French summaries)

Il s'agit de l'équation de la forme suivante:

$$Py'+y+(y-Ry')^2=0,$$

P et R étant des fonctions de x. Par la transformation de contact $x_1=\varphi y'$, $y_1=y-\varphi y'\int(1/\varphi)dx$ et au moyen de l'equation $x_1+y_1+y^2+k=0$ (k=constante) on obtient comme condition d'integrabilité: $R'P-RP'=P+R$.

Author's summary.

Sawyer, W. W. Differential equations with polynomial solutions. Quart. J. Math., Oxford Ser. 20, 22–30 (1949).

This paper establishes necessary and sufficient conditions for the differential equation

$$a(x)y''+b(x)y'+c(x)y=k(A(x)y''+B(x)y'+C(x)y)$$

to have a polynomial solution of exact degree n when the parameter k assumes the value k_n, for all nonnegative integers n. The procedure is to form a sequence of differential equations such that the nth equation is satisfied by the nth derivative of y. If, for $k=k_n$, this nth equation has a nonzero constant solution, then the original equation will have a polynomial solution. Clearly, we must have $c(x)=k_0C(x)$. With the notation $a_0=a/c$, etc.,

$$a_{n+1}=a_n/(b_n'+1),\quad b_{n+1}=(a_n'+b_n)/(b_n'+1),$$

and similarly for A_{n+1}, B_{n+1}, the author finds the set of conditions $k_na_n/A_n=k_0a_0/A_0$ for all n. Together with the condition on c, C these form a set of necessary and sufficient conditions in the "general case." The exceptional cases are also discussed [cf. the following review]. *A. Erdélyi.*

Chaundy, T. W. Differential equations with polynomial solutions. Quart. J. Math., Oxford Ser. 20, 105–120 (1949).

The author investigates linear ordinary differential equations $(F-kG)Y=0$ in which F and G are polynomials in x and $D=d/dx$ such that, for every nonnegative integer n, there is at least one value k_n of the constant parameter k for which a polynomial Y_n of exact degree n satisfies the differential equation. Such an equation is then said to be of Sawyer's type [cf. the preceding review]. It is shown that if $f(D,\delta)$ and $g(D,\delta)$ are polynomials with constant coefficients in D and $\delta=x(d/dx)$, then the equation $(\mathfrak{A})$: $(f-kg)Y=0$ is of Sawyer's type with $k_n=f(0,n)/g(0,n)$, provided that all these k_n are different from one another. From $(\mathfrak{A})$ a more involved form $(\mathfrak{A}_m)$ can be derived which has

$$F=D^m\left\{f(D,\delta)+\sum_{s=0}^{m-1}\lambda_sE_s(\delta)\frac{1}{\delta-s}\right\},$$

$$G=D^m\left\{g(D,\delta)+\sum_{s=0}^{m-1}E_s(\delta)\frac{1}{\delta-s}\right\},$$

where the $E_s(D)$ are arbitrary polynomials (with constant coefficients) in D, restricted only by the condition that they do not contribute to the terms of the highest order in the differential equation. A discussion of the problem aims to show that every equation of Sawyer's type is "reducible" (in an appropriate sense) to one of the form $(\mathfrak{A})$ or $(\mathfrak{A}_m)$. First and second order systems are discussed in greater detail. *A. Erdélyi* (Pasadena, Calif.).

***Bellman, Richard. A Survey of the Theory of the Boundedness, Stability, and Asymptotic Behavior of Solutions of Linear and Nonlinear Differential and Difference Equations.** Office of Naval Research, Washington, D. C., 1949. vi+156 pp.

Systems of ordinary differential equations, in addition to being of considerable intrinsic interest, are widely applied in various branches of theoretical physics and engineering. The subject has a vast literature, sometimes relatively inaccessible; and hence there is considerable duplication. It was hoped that a survey of the field would be a valuable book of reference and would stimulate research and prevent further duplication. The survey is concerned with the boundedness, stability and asymptotic behaviour of solutions of a system of real ordinary differential equations. In addition to the omission of a discussion of the solution in the complex plane, many other topics of considerable interest, such as the Sturm-Liouville theory, eigenfunction theory, existence and uniqueness theorems, topological methods, have had to be left out, either because they are adequately dealt with elsewhere or because such a branch would require a separate survey if it were to be adequately covered. Despite these limitations, the field covered is a large one and there is no room for proofs. Each chapter of the survey contains careful statements of the principal results with full explanations of the new ideas introduced

and references to the original papers in the bibliography at the end of the chapter.

The presentation is simplified by the use of matrix notation. The first chapter [54 pages] deals with the linear system $dz/dt=Az$, where A is an $n\times n$ matrix, z an n-dimensional column vector. The second [25 pages] is concerned with nonlinear systems, mainly with $dz/dt=A(t)z+f(z,t)$ where f is small compared with z as $z\to 0$. The third chapter [48 pages] is devoted principally to the linear second order differential equation. [We note in passing that the WKB method of the theoretical physicists is here called the Liouville-Jeffries (sic)-Wentzel-Kramers-Brillouin method. As Jeffreys himself has pointed out that the method was used by Carlini (1817) and Green (1837), one will soon need the whole alphabet.] In the last chapter [14 pages], the corresponding properties for the system of difference equations $z(t+1)=F(z(t),t)$ is considered. *E. T. Copson.*

*Sansone, Giovanni. **Equazioni Differenziali nel Campo Reale.** Vol. 2, 2d ed. Nicola Zanichelli, Bologna, 1949. xvi+475 pp. 4000 lire.

The first volume [2d ed., 1948] of this treatise has been reviewed previously [these Rev. **10**, 193]. In this concluding volume the treatment of ordinary differential equations in the real domain is continued in six chapters, devoted, respectively, to: the asymptotic properties of solutions; existence and uniqueness of solutions in the large; singular points and singular solutions; operational methods for solving linear equations; numerical, graphical, and mechanical methods of solution; discussions of some important particular differential equations.

In general the high standards which were set in the first volume have been fully maintained. Nevertheless, the qualities of the several chapters of the present volume are somewhat unequal. The first two chapters are especially noteworthy, because in them we find many important modern results which are not so conveniently accessible elsewhere. The final chapter is interesting and valuable, because it contains excellent discussions of some half dozen particular equations which are of importance in contemporary applied mathematics. The fourth and fifth chapters do not call for any special comment. The part of the third chapter that deals with singular points is disappointing, for it does not go beyond the mere rudiments of the subject. There are abundant references to the literature; and the volume is equipped with two good indices which contribute greatly to its usefulness. Typographically, the book leaves nothing to be desired.

In conclusion, it is appropriate to say a few words concerning the treatise as a whole. It seems to the reviewer that, in view of the general scope and nature of the work, the almost complete ignoring of the topological theory of differential equations (associated with the names of Poincaré and Bendixson) is a serious blemish. Perhaps also there should have been a chapter on Lie's group-theoretic treatment of differential equations. With these reservations, the reviewer considers this to be one of the most complete and valuable works in the field, and one which is to be highly recommended. *L. A. MacColl* (New York, N. Y.).

Sansone, Giovanni. Valutazione asintotica degli integrali dell'equazione di Liénard che per $t\to-\infty$ tendono allo zero. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 13–18 (1949).

The author discusses the solution of

$$x''+f(x)x'+x=0,$$

where $x'=dx/dt$, which tends to zero as $t\to-\infty$. Using the "variation of constants" formula with $f(x)$ replaced by $f(0)$ on the left and $(f(0)-f(x))x'$ on the right a number of results are obtained. [The author apparently does not realize that the well-known equivalent system

$$x'=-F(x)+y,\quad y'=-x,$$

where $F'(x)=f(x)$ and $F(0)=0$, is a Poincaré-Bendixson system. His results concern the situation at a simple singular point $(x=0, y=0)$ about which there is a tremendous literature where real variable hypotheses of the type he makes are exploited. The problem can also be regarded from the more general point of view of behaviour near a stable solution.] *N. Levinson* (Cambridge, Mass.).

Cartwright, Mary L., Copson, E. T., and Greig, J. Nonlinear vibrations. Advancement Sci. **6**, no. 21, 12 pp. (1949).

Expository article with an extensive bibliography.

Shimizu, Tatsujiro. On the existence of limit cycles for some nonlinear differential equations. Math. Japonicae **1**, 125–134 (1948).

The author obtains sufficient conditions on $f(x, y)$ and $\phi(x, y)$ for equations of the type $\ddot{x}+f(x,\dot{x})\dot{x}+\phi(x,\dot{x}^2)=0$ to have periodic solutions (which are not constant). The method generalizes that of the reviewer and Smith [Duke Math. J. **9**, 382–403 (1942); these Rev. **4**, 42].

N. Levinson (Cambridge, Mass.).

Barbuti, Ugo. Una proprietà che caratterizza l'unicità della soluzione delle equazioni differenziali ordinarie del primo ordine. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 298–303 (1949).

The author gives a necessary and sufficient condition that a solution $y=f(x)$ of the differential equation $y'=F(x, y)$ is unique. Although the condition itself is too detailed to state here, the proof is quite short, and the result may be easily shown to include known criteria for unicity due to Peano and Perron. *R. Bellman* (Stanford University, Calif.).

Naĭsul', A. B. Solution of a system of ordinary differential equations with initial data not of Cauchy type. Doklady Akad. Nauk SSSR (N.S.) **67**, 969–972 (1949). (Russian)

The author shows that the system

$$dy_i/dx=f_i(x, y_1, y_2, \cdots, y_n),\quad i=1, 2, \cdots, n,$$

has a unique set of solutions $y_i(x)$, $p\leqq x\leqq q$, satisfying the non-Cauchy initial conditions $y_i(x_i^0)=y_{i0}$, where $p\leqq x_i^0\leqq q$, providing that there is a region D: $p\leqq x\leqq q$, $|y_i-y_{i0}|\leqq b_i$, in which the following conditions hold. (I) The f_i are continuous, and the vector $f=(f_1, f_2, \cdots, f_n)$ satisfies a Lipschitz condition with respect to $y_1, y_2, \cdots, y_n$ with constant K, $Kn(q-p)<1$; (II) $\sup|f_i|=M_i<Knb_i$. The Picard method of successive approximation forms the basis of the proof, and yields the additional conclusion that the solution-vector $(y_1(x), y_2(x), \cdots, y_n(x))$ depends continuously upon the initial values x_i^0, y_{i0}, $i=1, \cdots, n$.

The condition $Kn(q-p)<1$ plays an essential role in the proof, and (perhaps in a weakened form) is necessary for the truth of the theorem. The author demonstrates this by considering the equation $y''+\lambda^2 y=0$, and the equivalent system $y_1'=y_2$, $y_2'=-\lambda^2 y_1$. Here $K=1$ if $\lambda\leqq 1$, $K=\lambda^2$ if $\lambda>1$. Direct calculation shows that if $x_2^0-x_1^0=\lambda^{-1}(\frac{1}{2}\pi+m\pi)$, m an integer, then the system has a solution only if $y_{10}+\lambda y_{20}=0$. The condition $2K(q-p)<\pi$ is necessary and sufficient to exclude this case. *J. G. Wendel.*

Colombo, Giuseppe. Sopra alcuni casi di riducibilità alle quadrature per le equazioni del moto di un punto sollecitato da forze posizionali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 286–290 (1949).

Differential equations of the type $d^2x_i/dt^2=X_i(x_1, \cdots, x_n)$, $i=1, 2, \cdots, n$ ($n=2$ and 3), are considered from the point of view of linear associative commutative algebras over the field of complex numbers. If $e_1, \cdots, e_n$ are the units in such a field the condition that $f(z)=\sum_{i=1}^n e_iX_i$ be a differentiable function of $z=\sum_{i=1}^n e_ix_i$ will enable us to write our system in the integrable form $d^2z/dt^2=g'(z)$. Thus by considering the various possible linear algebras of this type, several cases of integrability are obtained and classified.

D. C. Lewis (Baltimore, Md.).

Persidskiĭ, K. P. Uniform stability in the first approximation. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 229–240 (1949). (Russian)

The author considers the infinite system

$$dx_i/dt=f_i(x_1, x_2, \cdots, x_n, \cdots),$$

where

$$f_i=\sum_j p_{ij}x_j+g_i(x),$$

$|g_i(x)|\leqq\|x\|\,g(\|x\|)$ ($\|x\|=\sup_i(|x_1|, |x_2|, \cdots)$), $g(\|x\|)\to 0$ as $\|x\|\to 0$. Ordinary stability of the particular solution $x_1=x_2=\cdots=0$ requires the existence of an r, depending upon ϵ and in general t_0, such that $\|x(t_0)\|\leqq r$ implies $\|x(t)\|\leqq\epsilon$ for $t\geqq t_0$. If r is independent of t_0, the stability is said to be uniform. The stability is asymptotic if $\lim_{t\to\infty}x_i(t)=0$ for all i. Various theorems are demonstrated illustrating the connection between the uniform and asymptotic stability of the solution of the linear approximation and that of the solution of the full equation. In particular, a necessary and sufficient condition for uniform stability is given.

R. Bellman (Stanford University, Calif.).

Volk, I. M. On a sufficient condition for the stability of a motion in the critical case of two roots with vanishing real parts. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 459–462 (1949). (Russian)

Consider the autonomous system $dx/dt=X$, $dy/dt=Y$, $dX_\nu/dt=\sum p_{\nu j}x_j+X_\nu$, $k\neq 0$; $\nu, j=1, 2, \cdots, n$, where X, Y, X_ν are holomorphic in all the coordinates at the origin, the X_ν begin with terms of degree at least 2, and the characteristic roots of the matrix $\|p_{ij}\|$ have negative real parts. By a suitable change of the variables x_ν one may then dispose of the situation so that the lowest degree m of the terms in x, y alone of X, Y does not exceed the least of the same for the X_ν. These terms in X, Y are written $X^{(m)}$, $Y^{(m)}$ and if $m=1$ they are assumed to be $X^{(m)}=ky$, $Y^{(m)}=-kx$. The author proves the following theorem. If the form $xY^{(m)}-yX^{(m)}$ is definite and the system admits a holomorphic integral $H(x, y, x_1, \cdots, x_n)$ independent of t, such that

$$H(\epsilon x, \epsilon y, \epsilon^m x_1, \cdots, \epsilon^m x_n)/H(\epsilon x, \epsilon y, 0, \cdots, 0)\to 1$$

as $\epsilon\to 0$, and if furthermore the lowest degree terms in x, y alone in $H(x, y, \cdots)$ make up a definite form then: (a) the motion is stable near the origin; (b) if x_0, y_0 are small enough there is a periodic solution depending upon the two parameters $r_0=(x_0^2+y_0^2)^{\frac{1}{2}}$, $\theta_0=\tan y_0/x_0$, continuous in both x_0, y_0 and in t and reducing to the origin for $x_0=y_0=0$. Moreover when $m>1$ the period becomes infinite as $r_0\to 0$. There is also given an explicit determination of the period. [See I. M. Volk, Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] 10, 559–574 (1946); same journal 12, 647–650 (1948); these Rev. 8, 330; 10, 299].

S. Lefschetz.

Leighton, Walter. Principal quadratic functionals and self-adjoint second-order differential equations. Proc. Nat. Acad. Sci. U. S. A. 35, 192–193 (1949).

The functional $J(x)=\int_a^b[r(x)y'^2-p(x)y^2]dx$ is considered, where $r(x)$ and $p(x)$ are continuous and $r(x)>0$ for $0<x\leqq b$. A function $y(x)$ is called A-admissible if $y(x)$ is continuous on $0\leqq x\leqq b$, $y(0)=y(b)=0$, and if $y(x)$ is absolutely continuous and $y'^2(x)$ is in L on each closed subinterval of $0<x\leqq b$. Let $P(x)=\int_a^b p(x)dx$. Then if $p(x)>0$ for small x and if $P(0+)=\infty$, there exists an A-admissible $y(x)$ for which $J(0)$ is finite and negative. Another result is that if there is no point on $0\leqq x<b$ conjugate to $x=0$ and if $P(0+)<\infty$ then $J(0+)\geqq 0$ for all A-admissible $y(x)$. A third result relates $J(0+)\geqq 0$ to the nonexistence of focal points. Proofs are deferred to a later paper.

In the second part of the note two theorems on the oscillation of solutions of (*) $(r(x)y')'+p(x)y=0$ are stated. It is assumed that r, r' and p are continuous and $r>0$ on $0<x<\infty$. Theorem: if $p(x)>0$ near $x=+\infty$ and if(†) $\int_1^\infty\{r(x)\}^{-1}dx=\int_1^\infty p(x)dx=\infty$ then every solution of (*) has an infinite number of zeros as $x\to+\infty$. [Actually the condition $p(x)>0$ near $x=+\infty$ is not necessary.] The other oscillation theorem stated is for $(xy')'+p(x)y=0$. Theorem: if $p(x)>0$ and is $O(x^{-\alpha})$ ($\alpha>1$) for large x then $y(x)$ has at most a finite number of zeros for large x.

N. Levinson.

Wintner, Aurel. A priori Laplace transformations of linear differential equations. Amer. J. Math. 71, 587–594 (1949).

The equation (*) $x''+f(t)x=0$ is considered. For large t (*) is assumed to be nonoscillatory. Let the derivatives of $f(t)$ satisfy $(-1)^k f^{(k)}(t)\geqq 0$, $k=0, 1, 2, \cdots$, on $0<t<\infty$. Then there exists a T such that for any $t_0>T$ there is a solution of (*) in the form

$$x(t)=\exp\left[-\int_0^\infty(e^{-(t-t_0)s}-1)s^{-1}d\phi(s)\right],$$

where $\phi(s)$ depends on t_0, $d\phi(s)\geqq 0$, $\int_0^\infty e^{-\epsilon s}d\phi(s)<\infty$ for $\epsilon>0$ and $x(t_0)=0$.

N. Levinson (Cambridge, Mass.).

Wintner, Aurel. On almost free linear motions. Amer. J. Math. 71, 595–602 (1949).

Soit $f(t)$ une fonction continue pour $t_0<t<\infty$, avec $\int^\infty f(t)dt$ convergente et $\int^\infty(\max_{t\leqq\tau<\infty}|\int_\tau^\infty f(u)du|)dt<\infty$. En considérant les deux équations différentielles

$$x''(t)+f(t)x(t)=0, \quad y''(t)=0,$$

on peut établir entre leurs intégrales $x(t)$, $y(t)$ une correspondance (1, 1) de telle sorte que $x(t)\sim y(t)$ (pour $t\to\infty$).

A. Ghizzetti (Pisa).

Putnam, C. R. The cluster spectra of bounded potentials. Amer. J. Math. 71, 612–620 (1949).

Using the methods of Wintner [same J. 70, 22–30 (1948); these Rev. 9, 435] the author proves the following theorem concerning the set S' of cluster points of the spectrum of the eigenvalue problem

$$y''+(\lambda-q)y=0, \quad y(0)\cos\alpha+y'(0)\sin\alpha=0:$$

if q is continuous and bounded on the half line $(0, \infty)$, then every λ-interval, $\mu_1<\lambda<\mu_2$, contains at least one point of S' whenever the length of the interval satisfies

$$\mu_2-\mu_1>\limsup_{x\to\infty}q(x)-\liminf_{x\to\infty}q(x)$$

and $\mu_1\geqq\liminf_{x\to\infty}q(x)$.

M. J. Gottlieb (Chicago, Ill.).

Schäfke, Friedrich Wilhelm. Bemerkung über den Vergleich gewisser Eigenwertspektren. Math. Nachr. **1**, 292–294 (1948).

The author considers the effect on the eigenvalues of the problem $L(y)+(f_0(x)+\lambda g(x))y=0$ of replacing $f_0(x)$ by $f_0^*(x)$, the interval (a, b) and boundary conditions being unchanged; here $L(y)=\sum_{\nu=1}^{n}[f_\nu(x)y^{(\nu)}(x)]^{(\nu)}$, $f_n(x)\neq 0$, $g(x)>0$, and the $f_0^*(x)$, $f_\nu^{(\nu)}(x)$ and $g(x)$ are suitably continuous. It is proved that for every eigenvalue l of the original problem there are at least one, l^*, of the altered problem and a ξ in (a, b) such that $f_0(\xi)+lg(\xi)=f_0^*(\xi)+l^*g(\xi)$. The proof is based on the analytic dependence of λ on μ given by the eigenvalue problem

$$L(y)+((1-\mu)f_0(x)+\mu f_0^*(x)+\lambda g(x))y=0$$

with the same boundary conditions, and the relation

$$\lambda'(\mu)=-\int_a^b (f_0^*-f_0)y^2dx \Big/ \int_a^b gy^2dx.$$

A special case is noted: the eigenvalues of $y''+(\lambda-q(x))y=0$, $y(0)=y(1)=0$, with continuous $q(x)$ are given by

$$\lambda_\nu=\nu^2\pi^2+q(\xi_\nu),$$

$0\leqq\xi_\nu\leqq 1$. It is remarked that this can also be deduced from the oscillation and comparison theorems concerning the zeros. [Reviewer's remark. From this example it appears that l and l^* may be taken to correspond serially in the two sequences of eigenvalues, though this is not stated.]

F. V. Atkinson (Ibadan).

Groen, P. Exact solution of eigenvalue problems arising in connection with the study of certain hydrodynamical or quantum-mechanical wave phenomena. Appl. Sci. Research A. **1**, 225–236 (1948).

Author's summary: The differential equation

$$d^2\phi/dx^2+\{a[\cosh(x/c)]^{-2}+b[\sinh(x/c)]^{-2}-m^2\}\phi=0,$$

which is a special case of an equation that is encountered in various physical problems, is solved analytically. The eigenvalue problem which arises when this equation is combined with certain boundary conditions is discussed and solved for various cases.

N. Levinson (Cambridge, Mass.).

Touschek, B. Zum analytischen Verhalten Schrödinger'scher Wellenfunktionen. Z. Physik **125**, 293–297 (1949).

The one-dimensional Schroedinger equation for a two-particle system is considered:

$$(-d^2/dx^2+H_B+H'-E)\psi=0$$

where H_B is the Hamiltonian for particle B, $-d^2/dx^2$ that for particle A, H' is the interaction Hamiltonian, E is the energy. It is assumed that the problem $H_B\Phi_n=\epsilon_n\Phi_n$ has only two solutions Φ_1, Φ_2 with corresponding ϵ_1 and ϵ_2. Under the assumption $(\Phi_m, H'\Phi_n)=(1-\delta_{nm})\delta(x)\bar{K}$ (where $\delta(x)$ is the Dirac delta function), the Schroedinger equation is solved subject to the boundary conditions of a wave incident from the left of form e^{ikx}. The scattering matrix S may be easily found; in this case its elements are discontinuous. It is verified that S is unitary and the bound states of the total system are found from the imaginary poles of S.

H. Feshbach (Cambridge, Mass.).

Cherry, T. M. Uniform asymptotic expansions. J. London Math. Soc. **24**, 121–130 (1949).

The function $L(\mu v)$, where $\mu=(v^2-c)^{\frac{1}{2}}$ and $L(z)=z^{\frac{1}{2}}K_{\frac{1}{2}}(z)$, is a solution of

$$d^2y/dv^2+y(-v^2+c+\tfrac{3}{4}v^{-2})=0.$$

The substitution $v=\psi(u)$, $y=W\{\psi'(u)\}^{\frac{1}{2}}$ transforms this equation into $TW=0$, where

$$TW=\frac{d^2W}{du^2}+W\left\{(-\nu^2+c)\psi'^2+\frac{5\psi''^2}{36u^2}+\frac{\psi'''}{2\psi'}-\frac{3}{4}\left(\frac{\psi''}{\psi'}\right)^2\right\}.$$

The functions $\zeta^{\frac{1}{2}}J_\nu(\lambda(1-\zeta^2)^{\frac{1}{2}})$ and $\zeta^{\frac{1}{2}}Y_\nu(\lambda(1-\zeta^2)^{\frac{1}{2}})$ satisfy, if we put $u=\operatorname{arctanh}\zeta-\zeta$,

$$\frac{d^2w}{du^2}+w\left\{-\nu^2+\left(\frac{1}{\zeta^2}-1\right)\left(\frac{5}{4\zeta^4}-\frac{1}{4\zeta^2}+\lambda^2-\nu^2\right)\right\}=0,$$

and therefore also $Tw=wf(u)$, where

$$f(u)=(-\nu^2+c)\psi'^2+\frac{5\psi''^2}{36u^2}+\frac{\psi'''}{2\psi'}-\frac{3}{4}\left(\frac{\psi''}{\psi'}\right)^2+\nu^2-\left(\frac{1}{\zeta^2}-1\right)\left(\frac{5}{4\zeta^4}-\frac{1}{4\zeta^2}+\lambda^2-\nu^2\right).$$

By choosing c and $\psi(u)$ in a convenient manner, depending on n, the author obtains $f(u)=O(u^{-1}\nu^{-2n})$, for $|u|\leqq 1$; $f(u)=O(u^{-2}\nu^{-2n})$ for $|u|\geqq 1$, if $\lambda^2=\nu^2+\gamma_0+\gamma_1/\nu^2+\cdots$.

Starting with $L(\mu v)$ the method of Langer gives a solution

$$A\zeta^{\frac{1}{2}}J_\nu(\lambda(1-\zeta^2)^{\frac{1}{2}})+B\zeta^{\frac{1}{2}}Y_\nu(\lambda(1-\zeta^2)^{\frac{1}{2}})$$

of the differential equation $Tw=wf(u)$. Here $B=0$ and the author obtains thus, uniformly for $0\leqq\arg\nu\leqq\frac{1}{2}\pi-\epsilon$,

$$J_\nu(\lambda(1-\zeta^2)^{\frac{1}{2}})=\frac{L(\mu\psi(u))}{\pi\{\mu\zeta\psi'(u)\}^{\frac{1}{2}}}\{1+O(\nu^{-1-2n})\},$$

where ν is large and $\{\psi(u)-u\}u^{-1}\nu^2$ is a power series in $u^{\frac{1}{3}}$, of which each coefficient is a polynomial in $1/\nu^2$. This approximation to $J_\nu(\lambda x)$ involving $K_{\frac{1}{3}}(z)$ (or, if desired, Airy functions) is important, since it is uniform near the transition value 1 of x, where the function $J_\nu(\nu x)$ changes from monotonic to oscillating. The author gives an analogous approximation to $H_\nu^{(2)}(\lambda(1-\zeta^2)^{\frac{1}{2}})$ and he states that he can approximate by similar methods to the solution of any equation of the form

$$d^2w/du^2+u\{\nu^2f(u)+P(u, \nu^{-2})\}=0,$$

where $f(u)$ has a simple zero at $u=0$ and ν is large.

J. G. van der Corput (Amsterdam).

Segre, Beniamino. Trasporti rigidi di vettori, e geometria della retta. Ann. Mat. Pura Appl. (4) **27**, 267–272 (1948).

The problem is: to find necessary and sufficient conditions that the equations $x_ix_j'-x_jx_i'=p_{ij}(u)$ $(i, j=1, 2, \cdots, n)$ together with $\sum x_i^2=1$ have solutions $x_i=x_i(u)$. The $p_{ij}(u)$ are given functions of the real variable u. P. Nalli solved this problem for $n=3$ [Rend. Circ. Mat. Palermo **61**, 314–338 (1937)], and has recently done the same for $n=4$ [same Ann. (4) **26**, 67–72 (1947); these Rev. **10**, 149]. The present note solves the problem for any $n\geqq 3$, giving algebraic necessary and sufficient conditions involving only the p_{ij} and their first and second derivatives. The x_i are then easily and explicitly determined, generally without integration. Besides the application that the final result has to Nalli's researches on parallelism of vectors the various conditions found by the author have geometrical interpretation in the projective theory of ruled hypersurfaces.

J. L. Vanderslice (College Park, Md.).

Charles, Henri. Sur les systèmes de Pfaff linéaires homogènes à coefficients constants. Bull. Soc. Roy. Sci. Liége 17, 66–69 (1948).

Le système d'équations aux différentielles totales linéaires homogènes à coefficients constants:

$$dv=(a_1v+a_2w)dx+(b_1v+b_2w)dy,$$
$$dw=(c_1v+c_2w)dx+(d_1v+d_2w)dy$$

est équivalent au système d'équations

$$\partial v/\partial x=a_1v+a_2w,\quad \partial w/\partial x=c_1v+c_2w,$$
$$\partial v/\partial y=b_1v+b_2w,\quad \partial w/\partial y=d_1v+d_2w.$$

L'auteur se pose la tâche partant de ce système équivalent d'obtenir les résultats de Saltykow [C. R. Acad. Sci. Paris 225, 520–521 (1947); ces Rev. 9, 186] et Vessiot [ibid. 226, 289–291 (1948); ces Rev. 9, 354] en employant la transformation de Laplace. Le principe de la méthode de Saltykow consiste à chercher des solutions particulières de la forme exponentielle. Saltykow obtient ainsi la solution générale dans le cas particulier où certains déterminants ont toutes leurs racines simples. L'auteur développe une méthode, qui permet en outre de traiter le cas où les déterminants précités sont quelconques. *M. Pinl* (Cologne).

Charles, Henri. Sur l'intégrabilité des systèmes de Pfaff linéaires homogènes à coefficients constants. Bull. Soc. Roy. Sci. Liége 17, 155–157 (1948).

L'auteur traite le système d'équations aux différentielles totales linéaires homogènes à coefficients constants

$$dv=(a_1v+a_2w)dx+(b_1v+b_2w)dy,$$
$$dw=(c_1v+c_2w)dx+(d_1v+d_2w)dy.$$

Dans la note analysée ci-dessus l'auteur a discuté ce système en cas qu'il est complètement intégrable et il a calculé ses solutions. Desormais l'auteur se pose la tâche d'étudier ce système, sans qu'il soit nécessairement complètement intégrable. En employant la transformation de Laplace (comme dans la note précédente) plusieurs cas dépendants résultent et selon ceux-ci ou la seule solution triviale $v=w=0$ ou ∞^1 ou ∞^2 solutions du système. *M. Pinl* (Cologne).

Aržanyh, I. S. The invariant structure of differential equations under contact transformations. Doklady Akad. Nauk SSSR (N.S.) 66, 817–820 (1949). (Russian)

Generalizing some of his previous results [published in journals not circulated outside the USSR] the author proves the following theorem. Let $H_\pi, K_{\pi\rho}, H_{\pi\rho}, \cdots, H'_{\pi}, K'_{\pi\rho'}, H'_{\pi\rho'}$ ($\pi=1,\cdots,p$; $\rho=1,\cdots,r_\pi$; $\rho'=1,\cdots,r'_\pi$) be functions of $t_\pi; q_\pi, p_\pi; \cdots; q'_{\nu'}, p'_{\nu'}$ ($\nu=1,\cdots,n$; $\cdots$; $\nu'=1,\cdots,n'$) and let $F_{\pi\sigma}, G_{\pi\sigma}, \cdots, F'_{\pi\sigma'}, G'_{\pi\sigma'}$ ($\pi=1,\cdots,p$; $\sigma=1,\cdots,s_\pi$; $\sigma'=1,\cdots,s'_\pi$) be functions of the variables mentioned above and of the parameters $\vartheta^{(\alpha)}_{\pi\sigma}, \cdots, \vartheta'^{(\alpha')}_{\pi\sigma'}$ ($\alpha=1,\cdots a_{\pi\sigma}; \cdots$; $\alpha'=1,\cdots,a'_{\pi\sigma'}$), the parameters varying in domains $D_{\pi\sigma},\cdots,D'_{\pi\sigma'}$ which in general are dependent on the variables $t_\pi; q_\pi, p_\pi; \cdots; q'_{\nu'}, p'_{\nu'}$. Then the separated contact transformations

$$W=W(t_\pi; q_\nu, \xi_\nu),\quad W_\mu(t_\pi; q_\nu, \xi_\nu)=0,\quad \mu=1,\cdots,m,$$

$$p_\nu=\frac{\partial W}{\partial q_\nu}+\Lambda_\mu\frac{\partial W_\mu}{\partial q_\nu},\quad -\eta_\mu=\frac{\partial W}{\partial \xi_\nu}+\Lambda_\mu\frac{\partial W_\mu}{\partial \xi_\nu},\quad \mu=1,\cdots,m,$$

$$W'=W'(t_\pi; q'_{\nu'}, \xi'_{\nu'}),\quad W'_{\mu'}(t_\pi; q'_{\nu'}, \xi'_{\nu'})=0,\quad \mu'=1,\cdots,m',$$

$$p'_{\nu'}=\frac{\partial W'}{\partial q'_{\nu'}}+\Lambda'_{\mu'}\frac{\partial W'_{\mu'}}{\partial q'_{\nu'}},\quad -\eta'_{\nu'}=\frac{\partial W'}{\partial \xi'_{\nu'}}+\Lambda'_{\mu'}\frac{\partial W'_{\mu'}}{\partial \xi'_{\nu'}},$$

$$\mu'=1,\cdots,m'$$

transform the system of differential equations

$$dp_\nu+dt_\pi\left(\frac{\partial H_\pi}{\partial q_\nu}+K_{\pi\rho}\frac{\partial H_{\pi\rho}}{\partial q_\nu}+\int_{D_{\pi\sigma}}F_{\pi\sigma}\frac{\partial G_{\pi\sigma}}{\partial q_\nu}d\vartheta_{\pi\sigma}\right)=0,$$

$$dq_\nu-dt_\pi\left(\frac{\partial H_\pi}{\partial p_\nu}+K_{\pi\rho}\frac{\partial H_{\pi\rho}}{\partial p_\nu}+\int_{D_{\pi\sigma}}F_{\pi\sigma}\frac{\partial G_{\pi\sigma}}{\partial p_\nu}d\vartheta_{\pi\sigma}\right)=0,$$

$$\cdots\qquad\cdots$$

$$dp'_{\nu'}+dt_\pi\left(\frac{\partial H'_\pi}{\partial q'_{\nu'}}+K'_{\pi\rho'}\frac{\partial H'_{\pi\rho'}}{\partial q'_{\nu'}}+\int_{D'_{\pi\sigma'}}F'_{\pi\sigma'}\frac{\partial G'_{\pi\sigma'}}{\partial q'_{\nu'}}d\vartheta'_{\pi\sigma'}\right)=0,$$

$$dq'_{\nu'}-dt_\pi\left(\frac{\partial H'_\pi}{\partial p'_{\nu'}}+K'_{\pi\rho'}\frac{\partial H'_{\pi\rho'}}{\partial p'_{\nu'}}+\int_{D'_{\pi\sigma'}}F'_{\pi\sigma'}\frac{\partial G'_{\pi\sigma'}}{\partial p'_{\nu'}}d\vartheta'_{\pi\sigma'}\right)=0,$$

into the system of the same structure

$$d\eta_\nu+dt_\pi\left(\frac{\partial \tilde H_\pi}{\partial \xi_\nu}+\tilde K_{\pi\rho}\frac{\partial \tilde H_{\pi\rho}}{\partial \xi_\nu}+\int_{\tilde D_{\pi\sigma}}\tilde F_{\pi\sigma}\frac{\partial \tilde G_{\pi\sigma}}{\partial \xi_\nu}d\vartheta_{\pi\sigma}\right)=0,$$

$$d\xi_\nu-dt_\pi\left(\frac{\partial \tilde H_\pi}{\partial \eta_\nu}+\tilde K_{\pi\rho}\frac{\partial \tilde H_{\pi\rho}}{\partial \eta_\nu}+\int_{\tilde D_{\pi\sigma}}\tilde F_{\pi\sigma}\frac{\partial \tilde G_{\pi\sigma}}{\partial \eta_\nu}d\vartheta_{\pi\sigma}\right)=0,$$

$$\cdots\qquad\cdots$$

$$d\eta'_{\nu'}+dt_\pi\left(\frac{\partial \tilde H'_\pi}{\partial \xi'_{\nu'}}+\tilde K'_{\pi\rho'}\frac{\partial H'_{\pi\rho'}}{\partial \xi'_{\nu'}}+\int_{\tilde D'_{\pi\sigma'}}\tilde F'_{\pi\sigma'}\frac{\partial \tilde G'_{\pi\sigma'}}{\partial \xi'_{\nu'}}d\vartheta_{\pi\sigma'}\right)=0,$$

$$d\xi'_{\nu'}-dt_\pi\left(\frac{\partial \tilde H'_\pi}{\partial \eta'_{\nu'}}+\tilde K'_{\pi\rho'}\frac{\partial \tilde H'_{\pi\rho'}}{\partial \eta'_{\nu'}}+\int_{\tilde D'_{\pi\sigma'}}\tilde F'_{\pi\sigma'}\frac{\partial \tilde G'_{\pi\sigma'}}{\partial \eta'_{\nu'}}d\vartheta_{\pi\sigma'}\right)=0,$$

where

$$\tilde H_\pi=H_\pi+\frac{\partial W}{\partial t_\pi}+\Lambda_\mu\frac{\partial W_\mu}{\partial t_\pi},$$

$$\tilde H_{\pi\rho}=H_{\pi\rho},\quad \tilde K_{\pi\rho}=K_{\pi\rho},$$

$$\tilde F_{\pi\sigma}=F_{\pi\sigma},\quad \tilde G_{\pi\sigma}=G_{\pi\sigma},$$

$$\cdots\qquad\cdots$$

$$\tilde H'_\pi=H'_\pi+\frac{\partial W'}{\partial t_\pi}+\Lambda'_{\mu'}\frac{\partial W'_{\mu'}}{\partial t_\pi},$$

$$\tilde H'_{\pi\rho'}=H'_{\pi\rho'},\quad \tilde K'_{\pi\rho'}=K'_{\pi\rho'},$$

$$\tilde F'_{\pi\sigma'}=F'_{\pi\sigma'},\quad \tilde G'_{\pi\sigma'}=G'_{\pi\sigma'}$$

and $\tilde D_{\pi\sigma},\cdots,\tilde D'_{\pi\sigma'}$ are the transformed domains. It should be remarked that the summation sign is omitted if an index occurs twice in a product and that the notation $\int_{D_k}F(\vartheta)d\vartheta$ means

$$\int\int\cdots\int_{D_k}F(\vartheta^{(1)},\cdots,\vartheta^{(k)})d\vartheta^{(1)}\cdots d\vartheta^{(k)}.$$

E. Leimanis (Vancouver, B. C.).

Mitrinovitch, Dragoslav S. Remarque sur certaines équations aux dérivées partielles. Ist. Veneto Sci. Lett. Arti. Parte II. Cl. Sci. Mat. Nat. 99, 357–360 (1940).

Let $F_i(x_1,\cdots,x_n; z; p_1,\cdots,p_n)$ be a set of functions, and let $p_\nu=\partial z/\partial x_\nu$. The system of differential equations (*) $F_i=C_i$, where the C_i are constants, is said to be in involution if for each pair of F_i

$$\sum_{i=1}^{n}\left\{\frac{\partial F_k}{\partial p_i}\left(\frac{\partial F_j}{\partial x_i}+p_i\frac{\partial F_j}{\partial z}\right)-\frac{\partial F_j}{\partial p_i}\left(\frac{\partial F_k}{\partial x_i}+p_i\frac{\partial F_k}{\partial z}\right)\right\}=0.$$

Let $\varphi(x_1,\cdots,x_n; z; p_1,\cdots,p_n)=0$ be a differential equation that can be written in the form $\Phi(F_1,\cdots,F_k)=0$. The following remarks are made. If $k=n+1$, if $\Phi(C_1,\cdots,C_{n+1})=0$, and if the system $F_i=C_i$ ($i=1,\cdots,n+1$) is in involution,

then $F_i=C_i$ is a complete integral of $\Phi(F_1, \cdots, F_{n+1})=0$. If $k=n$, if $\Phi(C_1, \cdots, C_n)=0$, and if the system $F_i=C_i$ $(i=1, \cdots, n)$ is in involution, then on solving the $F_i=C_i$ for the p_k, one can use these p_k in $dz=\sum_{i=1}^{n}p_i dx_i$ to find a complete integral of $\Phi(F_1, \cdots, F_n)=0$. *F. G. Dressel.*

Amerio, Luigi. Su un metodo di integrazione delle equazioni differenziali lineari a derivate parziali. Rend. Sem. Mat. Fis. Milano **18** (1947), 114–123 (1948).

The author discusses the solution of boundary value problems associated with partial differential equations by a method suggested by Picone using the Fischer-Riesz theory.

F. G. Dressel (Durham, N. C.).

Amerio, Luigi. Relazioni tra il metodo della trasformata multipla di Laplace e il metodo di M. Riesz per l'integrazione di equazioni di tipo iperbolico. II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 48–52 (1949).

The author first generalises, for the equation

$$\left(\frac{\partial^2}{\partial x_1^2}-\frac{\partial^2}{\partial x_2^2}-\cdots-\frac{\partial^2}{\partial x_m^2}+\lambda\right)^k u=f,$$

a result obtained by M. Riesz for the case $k=1$, $\lambda=0$, viz., that

$$I^\alpha u(P)=\frac{1}{H_m(\alpha+2)}\int_{D_x^p} f(Q)r_{PQ}^{\alpha+2-m}dQ$$

$$+\frac{1}{H_m(\alpha+2)}\int_{S^p}\left\{\frac{\partial u(Q)}{\partial n}r_{PQ}^{\alpha+2-m}-u(Q)\frac{\partial r_{PQ}^{\alpha+2-m}}{\partial n}\right\}dS,$$

[Conférences Réunion Internat. Math., Paris, 1937, pp. 153–170 (1939), in particular, p. 158], where I^α denotes the generalised Riemann-Liouville integral. From this, Riesz found $u(P)=I^0u(P)$ by the method of analytical continuation.

In the present paper, $u(P)$ is found, not by analytical continuation, but by using the multiple Laplace transform discussed in an earlier paper by the same author [same Rend. Cl. Sci. Fis. Mat. Nat. (8) **5**, 313–319 (1948); these Rev. **10**, 610]. *E. T. Copson* (Dundee).

Amerio, Luigi. Relazioni tra il metodo della trasformata multipla di Laplace e il metodo di M. Riesz per l'integrazione di equazioni di tipo iperbolico. III. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 175–180 (1949).

[Cf. the preceding review.] The author finds the Laplace transform of $(x_1^2-x_2^2-x_3^2-\cdots-x_m^2)^{k\alpha}$ and the function whose Laplace transform is $(\lambda+p_1^2-p_2^2-p_3^2-\cdots-p_m^2)^{-k\beta}$.

E. T. Copson (Dundee).

Dronkers, J. J. An iterative process for the solution of a boundary value problem for a linear partial differential equation of the second order. I, II. Nederl. Akad. Wetensch., Proc. **52**, 329–337, 479–487 = Indagationes Math. **11**, 103–111, 139–147 (1949). (Dutch)

A function z is sought satisfying the equation

$$(1)\qquad \frac{\partial^2 z}{\partial x^2}+a\frac{\partial^2 z}{\partial y^2}+b\frac{\partial^2 z}{\partial x\partial y}+c\frac{\partial z}{\partial x}+d\frac{\partial z}{\partial y}+ez+f=0,$$

for $0\leqq x\leqq A$ and $0\leqq y\leqq B$, and such that z and $\partial z/\partial x$ assume given values when $x\to 0$. It is assumed that $a, b, \cdots, f$ represent functions which are continuous with respect to x in the given domain, that b and c have continuous derivatives $\partial b/\partial x$ and $\partial c/\partial x$, and that all these functions are analytic functions of y. Moreover it is required that for all these functions relations such as $|\partial^k a/\partial y^k|<M_1\rho^k k!$, where M_1 and ρ are constants, are valid for all values of k. The latter condition replaces the condition, often considered in analogous cases, that all these derivatives are bounded. It is to be remarked that no condition is introduced relative to the extension of the domain. The equation (1) is replaced by the set

$$(2)\qquad \begin{aligned}\frac{\partial z}{\partial x}&=\alpha\frac{\partial u}{\partial y}+\beta_1\frac{\partial z}{\partial y}+\gamma_1 z+\delta,\\ \frac{\partial u}{\partial x}&=\beta_2\frac{\partial z}{\partial y}+\gamma_2 z.\end{aligned}$$

The functions α, β_1, γ_1, δ, β_2, γ_2 are expressed in terms of $a, \cdots, f$. A solution of the set (2) is required such that $z\to z_0$ and $u\to u_0$ for $x\to 0$, where z_0 and u_0 are to be found from the given values for $z(0, y)$ and $z_x(0, y)$. The iterative process is as follows: z_1 and u_1 are found by replacing z and u in the second members of (2) by z_0 and u_0; z_n and u_n are found by replacing z and u in the same way by z_{n-1} and u_{n-1}. The uniform convergence of the process is proved by a method of majoration, where an estimation of the number of terms occurring in the expressions obtained by the successive integrations plays an important part.

H. Bremekamp (Delft).

Zwirner, Giuseppe. Su una particolare classe di equazioni alle derivate parziali del quarto ordine sopra una superficie chiusa. Rend. Sem. Mat. Univ. Padova **17**, 139–159 (1948).

Let S be a closed surface, topologically defined with an angular metric, introduced in the neighbourhood of any point by means of a local complex variable, yielding the conformal mapping of the neighbourhood into a bi-cell. The author applies a method due to Caccioppoli for the integration of the differential equation:

$$(1)\qquad L(u)\equiv\Delta[(\Delta u)/\alpha]+\beta u=g,$$

where Δ is the Laplacian and α, β indicate densities ($\alpha>0$). The idea of the method is the following. Since the differential form $L(u)dxdy$ is invariant with respect to the group of the conformal transformations of the neighbourhood of S, if E is a subset of S the additive set function

$$F(E)=\int_E\int L(u)dxdy$$

is intrinsically defined on S. Let us now consider the linear space Σ of all the additive functions

$$F(E)=\int_E\int \delta(P)dP$$

with density δ of summable $p/(p-1)$th power ($p>2$); after having normalized Σ in the following way:

$$\|F(E)\|=\left|\int_S\int[\delta(P)]^{p/(p-1)}dP\right|^{(p-1)/p},$$

let us consider the linear variety Σ^* of the $F(E)$'s, whose density is an $L(u)$. As one finds the variety Σ^* to be complete, it is therefore characterized by all the linear functionals not identically null in Σ, and null in Σ^*. But a linear functional is of the type $I(F)=\int\int s\delta v dP$, while on the other hand it may be shown (exploiting suitable average

proprieties which are characteristic for the solutions of $L(v)=0$) that, if $I(F)=0$ in Σ^*, v must be a solution of $L(v)=0$. Hence the following alternative theorem easily is deduced: if $L(v)=0$ has no autosolutions, (1) possesses solutions for any g; in the contrary case a necessary and sufficient condition that (1) has solutions is that g is orthogonal on S to all the autosolutions of $L(v)=0$, which are all linearly dependent on a finite number of them. *F. Conforto.*

Topolyanskiĭ, D. B. On the application of variational methods for the approximate solution of boundary problems for differential equations of elliptic type. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 317–320 (1949). (Russian)

Soit G un domaine plan, borné, limité par un contour L. Cherchons dans G la fonction $u(x, y)$, solution de l'équation du type elliptique $(pu_x)_x+(pu_y)_y=(q-a_x-b_y)u$ telle que sur L: $u=u_0(x, y)L$. Les fonctions a, b, p, q, u_0 sont données dans G. On admet que: (1) p, q, a, b, u_0 sont continues dans $G+L$; (2) $p>0$, $q\geqq 0$ sur $G+L$; (3) q admet des dérivées continues d'ordre 1, a et b d'ordre 2, p d'ordre 3.

R. Courant a réduit ce problème à la détermination de u rendant minimum la fonctionnelle:

$$E(u)=\int_G\int [p(u_x^2+u_y^2)+2(auu_x+buu_y)+qu^2]dxdy$$

sur un espace fonctionnel linéaire convenable, doué d'une métrique quadratique; on suppose que la forme quadratique correspondant à l'élément différentiel est définie (au sens de Courant). L'auteur aborde ce dernier problème par deux méthodes d'approximation dont une est celle de Ritz; il forme des fonctions u^{**} et u^* telles que: $E(u^{**})<E(u)<E(u^*)$ généralisant ainsi les résultats de Trefftz relatifs à l'équation de Laplace. Enfin, l'auteur illustre ses raisonnements par un exemple. *J. Kravtchenko* (Grenoble).

Ghizzetti, Aldo. Su un particolare problema misto per un'equazione di tipo ellittico a coefficienti costanti. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 5, 344–348 (1948).

The author considers the solution of the differential equation $A\zeta_{xx}+B\zeta_{yy}+C\zeta_x+D\zeta_y+E\zeta=0$ in the semi-infinite strip $0\leqq x\leqq l$, $y\geqq 0$, where A, B, C, D, E are constants, $AB>0$, subject to the conditions $\zeta(x, 0)=f(x)$, $f(x)$ continuous in $(0, l)$ and $\zeta_x(0, y)=\zeta_x(l, y)=0$. By the transformation $u=\zeta e^{\frac{1}{2}Cx/A+\frac{1}{2}Dy/B}$, $x\to xl/\pi$, $y\to(l/\pi)(B/A)^{\frac{1}{2}}y$, the problem becomes $u_{xx}+u_{yy}+\lambda u=0$ in $0\leqq x\leqq\pi$; $y\geqq 0$, with boundary conditions $u(x, 0)=e^{Clx/(2\pi A)}f(xl/\pi)$ and

$$u_x(0, y)-\alpha u(0, y)=u_x(\pi, y)-\alpha u(\pi, y)=0,$$

where $\alpha=lC/(2\pi A)$ and $\lambda=(l^2/\pi^2A)(E-\frac{1}{4}C^2/A-\frac{1}{4}D^2/B)$. The solution is found to be undetermined to the extent of a set of characteristic solutions

$$u_k(x, y)=\begin{cases}\varphi_k(x)\sin(\lambda-\mu_k)^{\frac{1}{2}}y, & k=0, 1, 2, \cdots, \nu-1;\\ \varphi_\nu(x)\sinh(\mu_\nu-\lambda)^{\frac{1}{2}}y, & k=\nu,\quad \lambda-\mu_\nu<0;\\ \varphi_\nu(x)y, & k=\nu,\quad \lambda-\mu_\nu=0;\\ \varphi_k(x)\sinh(\mu_k-\nu)^{\frac{1}{2}}y, & k=\nu+1, \nu+2, \cdots;\end{cases}$$

$$\varphi_0(x)=e^{\alpha x};\quad \varphi_k(x)=\alpha\sin kx+k\cos kx,\quad k=1, 2, 3, \cdots,$$

where $\mu_0=-\alpha^2$, $\mu_k=k^2$; $k=1, 2, 3, \cdots$ and ν is the least integer for which $\lambda-\mu_\nu\leqq 0$. The solution becomes determinate when $\nu=0$ or $\lambda+\alpha^2\leqq 0$ by restricting the limiting behavior for large values of y:

$$\lim_{y\to\infty} e^{-(\mu_\nu-\lambda)^{\frac{1}{2}}y}u(x, y)=0,\qquad \lambda-\mu_\nu<0;$$

$$\lim_{y\to\infty} u(x, y)/y=0,\qquad \lambda-\mu_\nu=0.$$

A uniqueness proof is not given. *F. E. Marble.*

Goheen, Harry E. A method for determining certain critical masses. J. Math. Physics 28, 107–116 (1949).

Consider a hot reacting substance surrounded by the same type of material but at a temperature too low to sustain the reaction. Let the temperature distribution $U(x, y, z; t)$ satisfy the equation

$$\partial U/\partial t=k\nabla^2U+\lambda\exp(-\epsilon/U),$$

where k, λ, ϵ are constants. Initially $U=U_0$ inside the hot substance and $U=U_1$ outside the reacting material. Let the shape of the hot core be that of a sphere with center at $(x, y, z)=(0, 0, 0)$ and radius α. The problem is to find a constant A such that if $\alpha>A$ then the function $U(0, 0, 0; t)$ increases monotonically, but if $\alpha<A$ the temperature $U(0, 0, 0; t)$ increases for a short period of time and then starts to decrease. In the present paper a method of approximating to the critical radius A is reduced to the problem of solving an infinite set of ordinary differential equations. The paper also treats similar problems in case the hot core is an infinitely long and wide slab or an infinitely long cylinder. *F. G. Dressel* (Durham, N. C.).

Integral Equations

Weyl, Hermann. Inequalities between the two kinds of eigenvalues of a linear transformation. Proc. Nat. Acad. Sci. U. S. A. 35, 408–411 (1949).

Let A be an $n\times n$ matrix; denote its eigenvalues by $\alpha_1, \cdots, \alpha_n$, and write $\lambda_i=|\alpha_i|^2$. Let the eigenvalues of $K=A^*A$ be $\kappa_1, \cdots, \kappa_n$, and suppose the λ's and κ's arranged so that $\lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n$ and $\kappa_1\geqq\kappa_2\geqq\cdots\geqq\kappa_n$. The author shows that if $\varphi(\lambda)$ is an increasing function for $\lambda>0$ such that $\varphi(e^\xi)$ is a convex function of ξ and $\varphi(0)=\lim_{\lambda\to 0}\varphi(\lambda)=0$, then

$$\varphi(\lambda_1)+\cdots+\varphi(\lambda_m)\leqq\varphi(\kappa_1)+\cdots+\varphi(\kappa_m)\qquad (m=1, 2, \cdots, n).$$

The proof begins by establishing the inequality

$$\lambda_1\lambda_2\cdots\lambda_m\leqq\kappa_1\kappa_2\cdots\kappa_m\qquad (m=1, 2, \cdots, n).$$

and the final result is deduced from a general lemma containing no reference to matrix theory.

The author then describes the application of the result to the theory of integral equations. In particular, it follows that if (α_i) are the eigenvalues (other than 0) of the equation $\int k(s, t)x(t)dt=\alpha x(s)$, arranged so that the $|\alpha_i|$ are in decreasing order, (σ_i) are the reciprocals of the singular values, similarly arranged, and $\sum\sigma_i^r$ is convergent for some $r>0$, then $\sum|\alpha_i|^r$ is convergent, and $\sum|\alpha_i|^r\leqq\sum\sigma_i^r$.

F. Smithies (Cambridge, England).

Zimmerberg, Hyman J. Definite integral systems. Duke Math. J. 15, 371–388 (1948).

The author considers a linear homogeneous vector integral equation

$$(1)\qquad y(x)=\lambda\int_a^b K(x, t)y(t)dt,\quad a\leqq x\leqq b,$$

where the elements of the matrix $K(x, t)$ are bounded functions whose discontinuities are regularly distributed on $a \leqq x, t \leqq b$, there is a matrix $S(x)$ such that: (H_0) $S(x)y(x) \not\equiv 0$ on $a \leqq x \leqq b$ if $y \not\equiv 0$ is a solution of (1) for some λ; (H_1) $S(x)$ is Hermitian with continuous elements on $a \leqq x \leqq b$; (H_2) $S(x)K(x, t) \equiv K^*(t, x)S(t)$, where K^* denotes the conjugate transpose of K, and, moreover, satisfying one of the following conditions of definiteness: (H_3) the Hermitian form $v^*S(x)v$ is nonnegative, $a \leqq x \leqq b$; $(H_{3,J})$ $\int_a^b \int_a^b g^*(x)K(x, t)g(t)\,dx\,dt \geqq 0$ for arbitrary vectors $g(x)$ with elements continuous on $a \leqq x \leqq b$. The results obtained extend those of Wilkins [same J. 11, 155–166 (1944); these Rev. 5, 267]. In particular, the class of equations (1) satisfying (H_0), (H_1), (H_2), $(H_{3,J})$ includes as a special case the vector integral equations equivalent to definite boundary value problems of the type considered by the reviewer [Trans. Amer. Math. Soc. 52, 381–425 (1942); these Rev. 4, 100]. *W. T. Reid* (Evanston, Ill.).

Germay, R.-H.-J. Sur des fonctions généralisant les noyaux itérés des équations intégrales de Volterra, de seconde espèce. Bull. Soc. Roy. Sci. Liége 16, 268–275 (1947).

In the integral equation of the Volterra type

$$\psi(x) = f(x) + \lambda \int_{x_0}^{x} K(x, s)\psi(s)ds$$

the author supposes that $f(x)$ is uniformly approximated by a sequence of functions $\{f_i(x)\}$ and that the kernel K is uniformly approximated by a sequence of kernels $\{K_i(x, y)\}$. He then considers the successive approximations to the solution in the form $\psi_i(x) = f_i(x) + \lambda \int_{x_0}^{x} K_i(x, s)\psi_{i-1}(s)ds$ and is thus led to a generalization of the iterated and resolvent kernels of the Volterra type. *I. A. Barnett.*

Germay, R.-H.-J. Sur la résolution des systèmes d'équations intégrales. Bull. Soc. Roy. Sci. Liége 17, 2–5 (1948).

The paper applies the Picard method of successive approximations to the system of integral equations of the Volterra type

$$\varphi_i(x) = f_i(x) + \lambda \int_a^x \sum_{j=1}^{3} K_{ij}(x, s)\varphi_i(s)ds \qquad (i = 1, 2, 3).$$

I. A. Barnett (Cincinnati, Ohio).

Germay, R.-H.-J. Sur une méthode d'approximations successives pour l'intégration des systèmes d'équations intégrales de seconde espèce de Volterra. Bull. Soc. Roy. Sci. Liége 17, 26–30 (1948).

The author again considers the same equations as in the preceding paper and obtains the $(\mu+1)$th approximating functions in the form

$$\varphi_r^{(\mu+1)}(x, \lambda) = f_r(x) + \lambda \int_a^x \sum_{i=1}^{3} K_{ri}^{(\mu+1)}(x, s; \lambda) f_i(s)ds \qquad (r = 1, 2, 3)$$

with recursion formulas for $K_{ri}^{(\mu+1)}(x, s; \lambda)$. He states without proof that these kernels have for limits the resolvents $\Gamma_{ri}(x, s; \lambda)$ of the original kernels $K_{ri}(x, s)$ as $\mu \to \infty$.

I. A. Barnett (Cincinnati, Ohio).

Chang, Shih-Hsun. A class of integro-differential equations. Amer. J. Math. 71, 563–573 (1949).

The author considers the class of nonlinear integro-differential equations $\partial\varphi(x, t)/\partial t = \int_a^b K(x, y)f(y, t, \varphi(y, t))dy$, and similar equations which are nonhomogeneous. If $\int_a^b K^2(x, y)dy$ is bounded and f is continuous in (x, t, φ) he shows that there exist solutions. If, furthermore, f satisfies a Lipschitz condition, then, under a certain initial condition, the solution is unique. The proof is based on Wintner's work on the theory of infinite systems of nonlinear differential equations. In the particular case $f = \varphi(x, t)$ the unique solution reduces to that found by Volterra who expressed the solution in terms of the iterated kernels of K. The solutions of the equations first mentioned are expressed in terms of the singular functions and singular values of the kernel and include in particular a solution for a general kernel similar to that found by the reviewer for symmetric and skew-symmetric kernels [Bull. Amer. Math. Soc. 26, 193–203 (1920)]. *I. A. Barnett* (Cincinnati, Ohio).

Functional Analysis, Ergodic Theory

Kantorovič, L. V. Functional analysis and applied mathematics. Vestnik Leningrad. Univ. 3, no. 6, 3–18 (1948). (Russian)

Cf. Uspehi Matem. Nauk (N.S.) 3, no. 6(28), 89–185 (1948); these Rev. 10, 380.

Zaanen, A. C. Note on a certain class of Banach spaces. Nederl. Akad. Wetensch., Proc. 52, 488–498 = Indagationes Math. 11, 148–158 (1949).

The author continues his previous investigations of L_Φ spaces [Ann. of Math. (2) 47, 654–666 (1946); these Rev. 8, 158; for definitions and notations see the cited review]. By defining the function $\varphi(u)$ to be nondecreasing and continuous on the left but not necessarily continuous and strictly increasing he includes the spaces L and L^∞, as well as L^p, $1 < p < \infty$, as special cases of L_Φ spaces. Various properties analogous to those of L^p spaces are established for L_Φ spaces. The case in which $\Phi(2u) \leqq C\Phi(u)$ for some $C > 0$ is of particular interest, the space being separable in this case. *R. E. Fullerton* (Madison, Wis.).

Wielandt, Helmut. Über die Unbeschränktheit der Operatoren der Quantenmechanik. Math. Ann. 121, 21 (1949).

The author gives a simple and straightforward proof that the equation $AB - BA = 1$ cannot hold for a pair of bounded operators A, B in a Banach space, where 1 denotes the identical operator. *F. Smithies* (Cambridge, England).

Višik, M. I. Linear extensions of operators and boundary conditions. Doklady Akad. Nauk SSSR (N.S.) 65, 433–436 (1949). (Russian)

Let A be a closed operator in a separable Hilbert space, $\mathfrak{H}$, with a domain $\mathfrak{D}(A)$ everywhere dense in $\mathfrak{H}$ and a bounded inverse. Let A' be a closed operator whose adjoint, A'^*, maps its domain onto $\mathfrak{H}$. By a proper extension $\bar{A}$ of A is meant a linear closed extension of A such that: (i) $A \subset \bar{A} \subset A'^*$, (ii) $\Re(\bar{A}) = \bar{A}\mathfrak{D}(\bar{A}) = \mathfrak{H}$, (iii) $\|\bar{A}^{-1}\| < \infty$. If U is the space of the solutions of $A^*u = 0$, V' the inverse image of U by A'^*, U' the space of solutions of $A'^* = 0$, and $V' = U' \oplus \bar{V}$, then the operator $\bar{A}$ coinciding with A'^* on $\mathfrak{D}(A) + \bar{V}$ is shown to be a proper extension. All the proper extensions $\bar{A}$ of A are described by the theorem: a necessary and sufficient condition for $\bar{A}$ to be a proper extension of A is that $\mathfrak{D}(\bar{A}) = \mathfrak{D}(A) + (\bar{A}^{-1} + B)U$, where B is a continuous operator mapping U on U', and that A coincides with A'^* on $\mathfrak{D}(A)$. Linear homogeneous "boundary

conditions" are described which are proper in the sense that they are satisfied by the functions of $\mathfrak{D}(A)$ and only by those. *M. J. Gottlieb* (Chicago, Ill.).

Višik, M. I. On linear boundary problems for differential equations. Doklady Akad. Nauk SSSR (N.S.) **65**, 785–788 (1949). (Russian)

The results of a previous article [see the preceding review] are applied to the Laplacian operator in a Hilbert space of L_2 functions in an n-dimensional region S with a sufficiently smooth boundary Γ. Let Δ_0 be the Laplacian defined in a domain consisting of twice continuously differentiable functions which vanish on a certain peripheral strip of S, Δ its closure, and Δ^* the adjoint of Δ. The existence of the Green's function of the Dirichlet problem for S assures the existence of a bounded inverse for Δ and hence of a linear extension $\bar{\Delta}$. The domain $\mathfrak{D}(\bar{\Delta})$ of any proper extension $\bar{\Delta}$ can according to the previous article be resolved into the linear direct sum:

$$\mathfrak{D}(\bar{\Delta})=\mathfrak{D}(\Delta)+(\bar{\Delta}^{-1}+B)U,$$

where U is the subspace of solutions u of $\Delta^*u=0$ and B is a continuous operator mapping U into itself. The main result is that the following boundary condition is proper: $\partial f(T)/\partial n_T=Rf(T)$, where T is a point of Γ, n_T the interior normal and R a positive self-adjoint differential operator. *M. J. Gottlieb* (Chicago, Ill.).

Nachbin, Leopoldo. A characterization of the normed vector ordered spaces of continuous functions over a compact space. Amer. J. Math. **71**, 701–705 (1949).

In order that a partially ordered normed linear space S be isomorphic to a subspace of the Banach lattice $c(E)$ of all real-valued continuous functions over a compact space E, it is necessary and sufficient that (i) every $x\in S$ is either semi-positive ($y\geqq x$ implies $\|y\|\geqq\|x\|$) or seminegative ($y\leqq x$ implies $\|y\|\geqq\|x\|$), (ii) the set of all positive elements ($x\geqq 0$) of S is closed. The proof is based on a generalization of the Hahn-Banach extension theorem to the case of ordered spaces. *S. Kakutani* (New Haven, Conn.).

Šilov, G. E. Rings of type C. Doklady Akad. Nauk SSSR (N.S.) **66**, 813–816 (1949). (Russian)

Šilov, G. E. Rings of type C on the line and on the circumference. Doklady Akad. Nauk SSSR (N.S.) **66**, 1063–1066 (1949). (Russian)

Let R be a commutative normed ring with identity and $\mathfrak{M}$ its space of maximal ideals topologized in the usual way and let $x\to x(M)$ be the homomorphism of R into the complex numbers associated with $M\in\mathfrak{M}$ [Gelfand, Rec. Math. [Mat. Sbornik] N.S. **9(51)**, 3–24 (1941); these Rev. **3**, 51]. The ring R is said to be "regular" provided, for closed $F\subset\mathfrak{M}$ and $M_0\notin F$, there exists $x\in R$ such that $x(M)=0$ for $M\in F$ and $x(M_0)\neq 0$. Let R be regular and, for $x\in R$, define $\|x\|_M=\inf\|y\|$, where the inf is taken over all $y\in R$ such that $y(M)$ coincides with $x(M)$ in some neighborhood of $M\in\mathfrak{M}$. If $\sup_M\|x\|_M$ gives a norm equivalent to $\|x\|$, then R is said to be of "type C" [Šilov, Trav. Inst. Math. Stekloff **21**, 1–118 (1947); these Rev. **9**, 596]. A ring (or ideal) is called "primary" if it contains (or is contained in) exactly one maximal ideal. Every maximal ideal M in a semi-simple regular ring R contains a minimal closed primary ideal [Šilov, ibid., p. 41] denoted by $J(M)$.

Consider a family $\{C_\alpha\}$ ($\alpha\in\Phi$) of circles C_α defined by $|t_\alpha|\leqq b_\alpha$ in the complex plane and form the bicompact product space $S=\prod_\alpha C_\alpha$. Let F be a closed subset of S and associate with each $t\in F$ a primary ring K_t. The set $\sum'_F K_t$ of all functions $x(t)$ defined on F, with $x(t)\in K_t$ and $\|x(\cdot)\|=\sup_t\|x(t)\|_t<\infty$, is a normed ring under the natural algebraic operations and $\|x(\cdot)\|$ as norm. Now, for $t=\{t_\alpha\}\in F$ associate with each α a particular element $z_\alpha(t)=x_{t_\alpha}+t_\alpha\cdot 1$ of $\sum_F K_t$, where x_{t_α} is in the maximal ideal of K_t. The closed subring of $\sum_F K_t$ generated by the z_α is called a "continuous direct sum of the first kind" and is denoted by $\sum_F K_t$. The closed subring of $\sum_F K_t$ generated by the exponentials $e^{\pm it_\alpha}$ is called a "continuous direct sum of the second kind" and is denoted by $\sum''_F K_t$.

Let R be an arbitrary ring of type C and denote by $\{z_\alpha\}$ a system of generators for R. Define C_α to be the set of t_α with $|t_\alpha|\leqq\max_M|z_\alpha(M)|$. Then the space $\mathfrak{M}$ of maximal ideals of R can be considered embedded in $S=\prod_\alpha C_\alpha$. For $t\in\mathfrak{M}\subset S$, set $K_t=R/J(t)$. Then $R=\sum'_{\mathfrak{M}}R/J(t)$. Conversely, if $R=\sum'_F K_t$ is a regular semi-simple ring, then R is of type C. Let $B=\prod_\alpha I_\alpha$, where I_α is the closed interval $[0, 2\pi]$ with 0 and 2π identified, and let R be a ring of type C of functions defined on B with the functions $z_\alpha(t)=e^{\pm it_\alpha}$ ($t=\{t_\alpha\}\in B$) as generators. Then $R=\sum''_B R/J(t)$. Conversely, if $R=\sum''_B K_t$ is a regular semi-simple ring, then R is of type C. Criteria are also given in the above two situations for regularity and semi-simplicity of the direct sums involved.

The second paper is concerned with applications of the above results in the following special cases. (1) The ring R is a "homogeneous ring on a circumference." That is, R is a ring of functions $x(t)$ defined on the closed interval $[0, 2\pi]$, with 0 and 2π identified, generated by $e^{\pm it}$ and such that $x(t+h)\in R$, $\|x(t+h)\|=\|x(t)\|$ for arbitrary real h. In this case all of the primary rings $R/J(t)$ are isomorphic. (2) The ring R is a ring of type C consisting of functions defined on the closed interval $[a, b]$, possessing one generator and containing all infinitely differentiable functions. Here the primary rings $R/J(t)$ are uniformly finite dimensional. (3) The ring R is a ring of type C on $[a, b]$ with one generator and containing all functions with continuous derivatives. A number of other concrete examples are also discussed. *C. E. Rickart* (New Haven, Conn.).

Halilov, Z. I. Linear singular equations in a normed ring. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 163–176 (1949). (Russian)

This paper is essentially an elaboration of results announced previously [Doklady Akad. Nauk SSSR (N.S.) **60**, 1133–1136 (1948); these Rev. **10**, 48]. *C. E. Rickart.*

Halmos, Paul R. A nonhomogeneous ergodic theorem. Trans. Amer. Math. Soc. **66**, 284–288 (1949).

The author discusses the convergence of the series (1) $\sum_{n=1}^{\infty}(f(T^nx)/n)$, where $f(x)$ is a real- or complex-valued measurable function and T is an ergodic measure-preserving transformation defined on a measure space (X, m) with $m(X)=1$. In case $f(x)\in L^1(X, m)$ with (2) $\int_X f(x)m(dx)=0$, the almost everywhere convergence of (1) was discussed by S. Izumi [Proc. Imp. Acad. Tokyo **15**, 189–192 (1939); these Rev. **1**, 148] under the assumption that T is mixing and that $m(T^nE\cap F)\to m(E)m(F)$ sufficiently fast and uniformly in E and F as $n\to\infty$. Here the author first shows that this condition of uniformity in E and F is too strong to be satisfied by any measure-preserving transformation unless the measure space (X, m) has essentially only two measurable sets: null sets and complements of null sets. Then the mean convergence of (1) for $f(x)\in L^2(X, m)$ is discussed. It is shown that the mean convergence holds on a

dense subset of $\mathfrak{N}$ but not on the whole set $\mathfrak{N}$, where $\mathfrak{N}$ is a closed linear manifold of $L^2(X, m)$ consisting of all $f(x) \in L^2(X, m)$ with (2). *S. Kakutani.*

Rohlin, V. A. On endomorphisms of compact commutative groups. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 329–340 (1949). (Russian)

An endomorphism of a measure space X with measure μ is a (not necessarily one-to-one) transformation T of X on itself such that, for every measurable set E, $T^{-1}E$ is measurable and $\mu(T^{-1}(E)) = \mu(E)$. (A nontrivial example is a continuous endomorphism of a compact group on itself.) In this paper the author (1) expounds the reviewer's results on automorphisms of compact Abelian groups [Bull. Amer. Math. Soc. **49**, 619–624 (1943); these Rev. **5**, 40], (2) defines a new measure-theoretic invariant of endomorphisms, avowedly designed to distinguish between spectrally equivalent automorphisms, and (3) proves, by a highly nontrivial algebraic technique, that the invariant is not strong enough to accomplish its original purpose. The new invariant is called r-fold mixing and the main theorem is that if an endomorphism of a compact Abelian group is nonperiodic (in the sense that the principal character of the group is the only one whose orbit is finite), then it is r-fold mixing. An endomorphism T is r-fold mixing if

$$\lim_{n\to\infty} \left(\bigcap_{i=0}^{r} T^{k_n^i} E_i \right) = \prod_{i=0}^{r} \mu(E_i)$$

whenever $E_0, E_1, \cdots, E_r$ are measurable sets and the sequence $\{(k_n^0, k_n^1, \cdots, k_n^r)\}$ consists of $(r+1)$-tuples of nonnegative integers with the property that

$$\lim_{n\to\infty} \min \{|k_n^i - k_n^j| : 0 \leqq i < j \leqq r\} = \infty.$$

For $r=1$, r-fold mixing reduces to the known concept of strong mixing. *P. R. Halmos* (Chicago, Ill.).

Rohlin, V. A. Selected topics from the metric theory of dynamical systems. Uspehi Matem. Nauk (N.S.) **4**, no. 2(30), 57–128 (1949). (Russian)

The purpose of this paper appears to be to describe to Russian readers the progress in the theory of measure-preserving transformations and flows in the last ten years. The paper contains no proofs other than those already available in the literature. Except for some of the recent modifications of the theory, due to the author, the results that are here expounded are those obtained by Ambrose, von Neumann, Kakutani, and the reviewer and have all been published in American journals. *P. R. Halmos.*

Theory of Probability

Bogolyubov, N. N., and Hacet, B. I. On some mathematical problems of the theory of statistical equilibrium. Doklady Akad. Nauk SSSR (N.S.) **66**, 321–324 (1949). (Russian)

Suppose that the potential energy W_s of s particles is the sum of mutual pairwise potentials $\Phi(r)$ of particles r units apart. Then the density of distribution of s particles if there are N in all is

$$F_s^{(N)} = c \int \cdots \int \exp\{-W_N/\theta\} dq_{s+1} \cdots dq_N,$$

where c is a constant and q_i stands for the three Cartesian coordinates of the ith particle. The following theorems are stated without proof. It is supposed throughout that $\Phi(r)$ is continuous except that $\Phi = +\infty$ near 0, vanishing at ∞ to an order higher than 3. When $N \to \infty$ the total volume V_N becomes infinite, with $v = V_N/N = \text{constant}$. (1) If

$$\frac{4\pi}{v} \int_0^\infty |e^{-\Phi(r)/\theta} - 1| r^2 dr < 1$$

then the sequence $F_s^{(1)}, F_s^{(2)}, \cdots$ is compact in terms of uniform convergence on bounded $q_1, \cdots, q_s$ sets. (2) Under slightly stronger regularity conditions on Φ any limit density satisfies a certain explicitly written differentio-integral equation. (3) If $\Phi(r) \geqq 0$ for all r, and satisfies certain further regularity conditions, and if v is sufficiently large, the sequence $\{F_s^{(N)}\}$ converges for all s, and satisfies a functional equation in s and the q's [the latter equation has been derived formally by Mayer and Montroll, J. Chem. Phys. **9**, 2–16 (1941)]. *J. L. Doob* (Ithaca, N. Y.).

Fréchet, Maurice. Les valeurs typiques d'ordre nul ou infini d'un nombre aléatoire. Rev. Inst. Internat. Statistique **16**, 1–22 (1948).

A typical value of order α of a random variable X is a value of a minimizing $E\{|X-a|^\alpha\}$. The author investigates the various possible extensions of this concept for $\alpha = 0$ and $\alpha = \infty$. [He treats the case of abstract-valued random variables in Ann. Sci. École Norm. Sup. (3) **65**, 211–237 (1948); these Rev. **10**, 386.] *J. L. Doob.*

Kac, M. On the average number of real roots of a random algebraic equation. II. Proc. London Math. Soc. (2) **50**, 390–408 (1949).

Let $X_0 + X_1 x + \cdots + X_{n-1} x^{n-1} = 0$ be an algebraic equation whose coefficients $X_0, X_1, \cdots, X_{n-1}$ are random variables with distribution function $\sigma(u)$. The author gives an expansion for the average number of roots of the above equation in the interval (a, b). He then considers the special case when $\sigma(u) = \frac{1}{2}u$, and deduces from the general formula that in this case the average number of real roots is asymptotic to $(2\pi)^{-1} \log n$. This estimate is the same as that obtained by the author in part I of this paper [Bull. Amer. Math. Soc. **49**, 314–320 (1943); these Rev. **4**, 196], for the case of normally distributed X's. It is an improvement in a result of Littlewood and Offord [Proc. Cambridge Philos. Soc. **35**, 133–148 (1939)] who showed that the average number could not exceed $A(\log n)^2$. *A. C. Offord.*

Erdös, P. On a theorem of Hsu and Robbins. Ann. Math. Statistics **20**, 286–291 (1949).

Let $X_1, X_2, \cdots$ be independent and identically distributed random variables. It is proved that a necessary and sufficient condition for the convergence of $\sum_{n=1}^{\infty} P\{|\sum_{k=1}^{n} X_k| > n\}$ is that $|EX_1| < 1$ and $EX_1^2 < \infty$ hold. The sufficiency part is the Hsu-Robbins theorem; the proof given is simpler than theirs. [The author writes in a letter that a conjecture, attributed to Hsu and Robbins and stated to be inaccurate, is misquoted in his paper; correctly stated it is that, if $EX_1 = 0$ and $\sum_{n=1}^{\infty} P\{|\sum_{k=1}^{n} X_k| > n\} < \infty$, then $EX_1^2 < \infty$; and this is true.] *M. Loève* (Berkeley, Calif.).

Sapogov, N. A. A two-dimensional limit theorem for two-dimensional chains. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 301–314 (1949). (Russian)

The author proves the validity of the central limit theorem for two-dimensional nonstationary Markov chains (that is, that the appropriate pairs of random variables have approxi-

mately a bivariate normal distribution). He uses Bernstein's classical theorems on the central limit theorem for dependent random variables, but his hypotheses are too complicated to be stated here. *J. L. Doob.*

Otter, Richard. The multiplicative process. Ann. Math. Statistics **20**, 206–224 (1949).

The author investigates processes, sometimes called birth processes, or stationary branching processes, in which an individual μ_0 gives birth to W individuals $\mu_1, \mu_2, \cdots$, where $W=j$ with probability p_j; μ_j in turn produces further individuals $\mu_{j1}, \mu_{j2}, \cdots$, and so on. Births are mutually independent and all individuals have the same distribution of offspring. The study is first put on a rigorous basis by considering the individuals as vertices of a tree and setting up a measure in the space of trees [cf. also Everett and Ulam, Proc. Nat. Acad. Sci. U. S. A. **34**, 403–405 (1948); these Rev. **10**, 132]. The following are among the results obtained. (1) If P_n is the probability that the tree will be finite and have n vertices, $P_n \sim c^{-n}n^{-\frac{3}{2}}$ for large n, where c is a constant (here $n \equiv 1 \pmod q$), where q is the largest integer such that $p_j \neq 0$ implies that j is a multiple of q; for other values of n, $P_n=0$. (2) An asymptotic expression (n large) is found for the probability that $W=j$ if the tree is finite with n vertices. (3) The distributions of the fraction of vertices with k outgoing segments in trees which are finite, and which have n vertices, are found, as well as an asymptotic formula for large n in the latter case.

J. L. Doob (Ithaca, N. Y.).

Yosida, Kôsaku. Simple Markoff process with a locally compact phase space. Math. Japonicae **1**, 99–103 (1948).

Let X be a metric space whose bounded closed subsets are compact and let $P(x, A)$ be a transition probability defining a Markov process, so that $P(x, A)$ is the probability of a transition from x in X to a point of the Borel set $A \subset X$. Let C be the class of real continuous functions $f(x)$ with the property that the closure of the set $\{x, f(x) \neq 0\}$ is compact. It is supposed that if $f \in C$ then $\int_X f(y)P(x, dy) \in C$ also. Under these hypotheses it is shown that there is a decomposition of X into ergodic and dissipative parts, and that in each ergodic part the transition probability becomes a transition probability in a (smaller) space with an essentially unique stable probability distribution. For the case of a compact state space, see the author [Proc. Imp. Acad. Tokyo **16**, 43–48 (1940); these Rev. **1**, 343] and Beboutoff [Rec. Math. [Mat. Sbornik] N.S. **10**(52), 213–238 (1942); these Rev. **7**, 19]. *J. L. Doob* (Ithaca, N. Y.).

Yaglom, A. M. On the statistical reversibility of Brownian motion. Mat. Sbornik N.S. **24**(66), 457–492 (1949). (Russian)

Further details and proofs of material discussed in an earlier paper [Doklady Akad. Nauk SSSR (N.S.) **56**, 691–694 (1947); these Rev. **9**, 150]. *J. L. Doob.*

Mathematical Statistics

McCarthy, Philip J. Approximate solutions for means and variances in a certain class of box problems. Ann. Math. Statistics **18**, 349–383 (1947).

Consider n boxes associated with probabilities p_i ($\sum p_i = 1$). If the ith box is considered admissible when at least k_i balls are in it, let $N_s[k_1(p_1), k_2(p_2), \cdots, k_n(p_n)]$ represent the number of balls which have been thrown one by one into the boxes at the time s boxes have become admissible. Then N_s is a random variable for which the author indicates a method of computing the moments. Exact moments are given for $N_1[k_1(p_1), k_2(p_2)]$, $N_2[k_1(p_1), k_2(p_2)]$. Because of the great labor involved for $n>2$, approximations are given to (1) the mean and variance for $s=1$ and $s=n$ and (2) the mean for the cases where $2 \leqq s \leqq n-1$, $p_i=1/n$, $k_i=k$. Indications are given concerning the error of the approximations. Applications are mentioned to problems in sequential analysis, random walk, public opinion polling. *H. Chernoff.*

Banerjee, D. P. On percentage points of incomplete beta-functions and χ^2 distribution. Bull. Calcutta Math. Soc. **41**, 53–54 (1949).

The first terms of Taylor series around $x=0.5$ [$x=1$] are given. No attention is paid to convergence or to the number of terms required for a useful approximation.

J. W. Tukey (Princeton, N. J.).

Birnbaum, Z. W., and Andrews, F. C. On sums of symmetrically truncated normal random variables. Ann. Math. Statistics **20**, 458–461 (1949).

An approximate solution is given for the following problem which arises in quality control work. Given A and ϵ, determine k so that sums S_n of n observations from a symmetrically truncated normal distribution with range $2k$ will satisfy $P(|S_n| \geqq A) = \epsilon$. *A. M. Mood.*

Hatke, Mary Agnes. A certain cumulative probability function. Ann. Math. Statistics **20**, 461–463 (1949).

The author continues the investigation of the cumulative probability function $F(x) = 1-(1+x^{1/c})^{-1/k}$, $0 \leqq x < \infty$, initiated by Burr [same Ann. **13**, 215–232 (1942); these Rev. **4**, 19]. Tables are given from which the parameters c and k are readily determined by means of the observed values of a_3 and a_4, the measures of skewness and kurtosis. The region of these curves is determined on the Craig $\delta - \alpha_3^2$ chart. The chief advantage in the graduation by $F(x)$ is the substitution of differencing for integration. *L. A. Aroian.*

Riordan, John. Inversion formulas in normal variable mapping. Ann. Math. Statistics **20**, 417–425 (1949).

Let $P(a, c) = \sum_c^\infty e^{-a}a^x/x!$, the probability of obtaining c or more successes in the Poisson probability function. Define g by $P(a, c) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{g} e^{-x^2/2}dx$. Campbell [Bell System Tech. J. **2**, no. 1, 95–113 (1923)] showed essentially, if $x=(a-c)c^{-\frac{1}{2}}$, that $x \sim g+(g^2-1)c^{-\frac{1}{2}}/3+(g^3-7g)c^{-1}/36+\cdots$. This result may be inverted to express g in terms of x and c. The author investigates the problem in general for one-parameter probability functions, and determines methods for obtaining either series from the other. Two theorems whose precise formulations are long are proved. The first states that if $G_1(g), G_2(g), \cdots$ are assigned polynomials and if $x = g+\sum_{n=1}^\infty G_n(g)y^n/n!$ defines x in terms of g and a parameter y, then $g = x+\sum_{n=1}^\infty X_n(x)y^n/n!$ where $-X_n(x)$ is defined in terms of the polynomials of Bell [Ann. of Math. (2) **35**, 258–277 (1934)]. The second theorem relates to expansions of the form $a = c\sum_0^\infty q_n(g)c^{-\frac{1}{2}n}/n!$ and the inverse expression $c = a\sum_0^\infty p_n(g)a^{-\frac{1}{2}n}/n!$, where $q_i(g)$ are given polynomials, and $p_n(g)$ are expressible as Bell polynomials. The applications of the first theorem result in well-known series for the χ^2 and Student's t distributions. *L. A. Aroian.*

Richter, H. Zur Maximalkorrelation. Z. Angew. Math. Mech. **29**, 127 (1949).

Friede and Manzer [same Z. **28**, 158–160 (1948); these Rev. **10**, 50] showed that if the correlation [contingency] table of two variates is reducible as a matrix, then their "maximal correlation" is unity. The author shows the converse. *J. W. Tukey* (Princeton, N. J.).

Seal, H. L. A note on the χ^2 smooth test. Biometrika **35**, 202 (1948).

The note concerns the combination of the χ^2 test of goodness of fit with a test involving sequences of signs of the deviations of the observed frequencies from their hypothetical means [cf. F. N. David, Biometrika **34**, 299–310 (1947); these Rev. **9**, 600]. A theorem is given which can be paraphrased as follows. If $x_1, \cdots, x_n$ are n independent random variables normally distributed about zero mean with unit variance, then the conditional distribution of $q^2=x_1^2+\cdots+x_n^2$ under the condition that the x's satisfy k independent homogeneous linear relations is not changed if the additional condition is imposed that the x's are different from 0 and their signs are fixed in an arbitrary way compatible with the linear relations. *W. Hoeffding.*

Finney, D. J. The truncated binomial distribution. Ann. Eugenics **14**, 319–328 (1949).

In problems where observations are from binomial populations but an observation must be at least one to be recorded one must consider the truncated binomial distribution. An iterative method is given for obtaining the maximum likelihood estimates for p. Tables are constructed so that this method requires very little work. In the examples considered good convergence is obtained in one or two iterations. The iterative method is a gradient method of maximization which is asymptotically equivalent to the Newton method. It is also applied to the case where double truncation occurs, i.e., observations must be at least two to be recorded. *H. Chernoff* (Urbana, Ill.).

Fréchet, Maurice. Sur l'estimation statistique. Ann. Soc Polon. Math. **21** (1948), 207–213 (1949).

Let P be the parameter of a binomial distribution. When the a priori probability distribution of P is such that the probability is small (in a precisely defined sense) that P lie near 0 or 1, the author constructs "intervals of security" whose a posteriori probability of containing P is not less than α, with α a specified number less than 1.

J. Wolfowitz (New York, N. Y.).

Roy, S. N. Notes on testing of composite hypotheses. II. Sankhyā **9**, 19–38 (1948).

Let the probability density of a random vector

$$\mathbf{X}=(X_1, \cdots, X_n)$$

be of the form $\phi(\mathbf{x}, \theta, \tau)=\phi_1(F_1, \cdots, F_m, \theta, \tau)\phi_2(\mathbf{x}, \tau)$, where θ and τ are two parameter vectors, $F_i=F_i(\mathbf{x}, \tau)$ are independent of θ, $m<n$. The hypothesis to be tested is $\tau=\tau^0$ (θ unspecified). In a previous paper [Sankhyā **8**, 257–270 (1947); these Rev. **9**, 454] the author determined a family of similar regions based on the functions F_i (analogous to Neyman's similar regions based on sufficient statistics). In the present paper that similar region of this family is determined which is (uniformly) most powerful with respect to the set of alternatives $\tau=\tau^1$ (θ unspecified). This is done for the cases (i) the F_i are independent of τ; (ii) $F_1, \cdots, F_m$ and ϕ_2 are independent of τ, with additional conditions ensuring that the boundary of the region is independent of θ. [A third case discussed by the author is actually a special case of (ii).] Conditions are given for the test to be uniformly most powerful against a set of alternatives with varying τ. Familiar examples illustrate the method.

W. Hoeffding (Chapel Hill, N. C.).

Hansen, Morris H., and Hurwitz, William N. On the determination of optimum probabilities in sampling. Ann. Math. Statistics **20**, 426–432 (1949).

Suppose that a population contains M "primary units," m of these units are sampled with replacements (a mathematical convenience) and if the ith unit is in the sample, n_i elements are drawn from it and two characteristics are measured. One is to estimate

$$X/Y=\sum_{i=1}^{M}\sum_{j=1}^{N_i}X_{ij}\Big/\sum_{i=1}^{M}\sum_{j=1}^{N_i}Y_{ij}.$$

It is desired to minimize the variance of the estimate of X/Y subject to a fixed cost and $n_iP_i/N_i=k$, where N_i is the size of the ith primary unit, P_i the probability of selecting the ith primary unit and k constant. Optimal values of P_i, k and m are obtained in terms of the N_i and of the parameters of the cost function and of the primary units. [Bars should be omitted from X and Y on the top of page 28.] *H. Chernoff* (Urbana, Ill.).

Halmos, Paul R., and Savage, L. J. Application of the Radon-Nikodym theorem to the theory of sufficient statistics. Ann. Math. Statistics **20**, 225–241 (1949).

A set M of probability measures on a measurable space X is said to be dominated if there exists a measure λ on X (λ is not necessarily a member of M) such that every member of M is absolutely continuous with respect to λ. Using the Radon-Nikodym theorem, the authors obtain necessary and sufficient conditions for a statistic T (point function on X) to be sufficient for a dominated set M of measures. The main result is a generalization of the well-known Fisher-Neyman theorem to the abstract case. The authors consider also the notion of pairwise sufficiency. A statistic T is said to be pairwise sufficient for a set M of measures if it is sufficient for every pair of measures in M. Necessary and sufficient conditions are given for a statistic T to be pairwise sufficient for a set M of measures. The authors show that the concepts of pairwise sufficiency and sufficiency coincide when M is dominated, but not necessarily so when M is not dominated. *A. Wald* (New York, N. Y.).

Kosambi, D. D. Characteristic properties of series distributions. Proc. Nat. Inst. Sci. India **15**, 109–113 (1949).

For $a_n \geqq 0$ consider the series $\sum_0^\infty a_n z^n=f(z)$ and let x be a random variable with $\Pr\{x=n\}=a_n z^n/f(z)$. Here z is a positive parameter. The moments and factorial moments of the distribution are easily calculated in terms of $f(z)$ and simple differential operations performed on $f(z)$. If a random variable depending on the parameter z has a distribution of the above type with mean and variance equal for all positive z, then $f=e^z$ and the distribution is the Poisson distribution with mean z. The maximum likelihood estimate of z from an observed sample is obtained by equating the observed and theoretical means. Various remarks are made to justify the wide application of distributions determined in this way. *J. L. Doob* (Ithaca, N. Y.).

Doob, J. L. Heuristic approach to the Kolmogorov-Smirnov theorems. Ann. Math. Statistics **20**, 393–403 (1949).

Let $\{\rho(t)\}$, $0 \leqq t \leqq \infty$, be a Wiener stochastic process, i.e.: (a) the distribution of any finite number of the chance variables is Gaussian, with all expected values zero; (b) we have $E\{x(s)x(t)\} = \min(s, t)$. The author proves that, if $a \geqq 0$, $b > 0$, $\alpha \geqq 0$, $\beta > 0$, then

$$(1) \qquad P\left\{\sup_{0 \leqq t < \infty} [\rho(t) - (at+b)] \geqq 0\right\} = e^{-2ab},$$

$$(2) \quad P\left\{\sup_{0 \leqq t < \infty} [\rho(t) - (at+b)] \geqq 0 \text{ or } \inf_{0 \leqq t < \infty} [\rho(t) + \alpha t + \beta] \leqq 0\right\}$$

$$= \sum_{m=1}^{\infty} \exp[-2\{(m^2ab + (m-1)^2\alpha\beta + m(m-1)(a\beta+\alpha b)\}]$$

$$+ \sum_{m=1}^{\infty} \exp[-2\{(m-1)^2ab + m^2\alpha\beta + m(m-1)(a\beta+\alpha b)\}]$$

$$- \sum_{m=1}^{\infty} \exp[-2\{m^2(ab+\alpha\beta) + m(m-1)a\beta + m(m+1)\alpha b\}]$$

$$- \sum_{m=1}^{\infty} \exp[-2\{m^2(ab+\alpha\beta) + m(m+1)a\beta + m(m-1)\alpha b\}].$$

The first of the above results and the second one, when suitably specialized, are suggestive of the results obtained by Kolmogorov [Giorn. Ist. Ital. Attuari **4**, 83–91 (1933)] on the limit of the distribution functions of certain functions of the differences between the "true" distribution function and the "sample" distribution function, and of some of the results obtained by Smirnov [Rec. Math. [Mat. Sbornik] N.S. **6(48)**, 3–26 (1939); Bull. Math. Univ. Moscow **2**, no. 2 (1939); these Rev. **1**, 246, 345] on the limit of the distribution function of the maximum absolute difference between two sample distribution functions. Indeed, these latter results follow from the author's results if one allows a certain very suggestive inversion of two limiting processes. This inversion is not legitimized in the present paper, and this explains the title. [The reviewer notes the following errors: In the first displayed equation at the top of page 402, u should be replaced by x. In the statement of (4.3) it is already assumed that $\sigma = 1$.] *J. Wolfowitz.*

Scheffé, Henry. Operating characteristics of average and range charts. Industrial Quality Control **5**, 13–18 (1949).

Address to the Institute of Mathematical Statistics and the American Society for Quality Control.

Tukey, John W. Comparing individual means in the analysis of variance. Biometrics **5**, 99–114 (1949).

Since the F-test cannot be used for solving the problem of breaking up the "treatment" means into distinguishable groups, the author recommends the following procedure. (1) Break up the means into one or more broad groups by using, repeatedly, a test of excessive gapping between adjacent means. (2) Separate off, within each group, any mean which straggles too much from the mean of the group by using, repeatedly, the Studentized extreme deviate $u = (m_1 - \bar{m})/s_m$, where m_1 is the most straggling mean, $\bar{m}$ the group mean, and s_m^2 the estimated variance of a single mean, based on n degrees of freedom. (3) Apply the F-test to each of the resulting groups to detect excessive variability.

As a conservative test of gapping the author proposes the ratio (largest gap between adjacent means)/$(s_m 2^{\frac{1}{2}})$, the 5%, 2%, and 1% points of which, according to experimental sampling performed by the author, are smaller than those of the corresponding two-sided percentage points of the t-distribution with n degrees of freedom. Since, for large G, the expected total number p_1 of gaps longer than G per sample of size k from a normal population with mean 0 and variance 1 differs by a small amount from the expected number of largest gaps longer than G per sample, p_1 has been used for checking the experimental results. For more general p_1 the following expressions are derived:

$$p_1 = k\int_{-\infty}^{+\infty} \{(F(y)+1-F(y+G))^{k-1} - (F(y))^{k-1}\}dF(y),$$

when $F(y)$ is the cumulative distribution function of an arbitrary population sampled from, and

$$p_1 = 2k\int_{-\infty}^{-\frac{1}{2}G} \{(F(y)+1-F(y+G))^{k-1} - (2f(y))^{k-1}\}dF(y)$$

in the special case when the population is distributed symmetrically about zero.

The upper percentage points of u are, according to further experimental sampling, said to be obtained with good approximation by treating $\frac{1}{3}(u - \frac{8}{3}\log_{10} k)/(\frac{1}{4} + 1/n)$ for $k > 3$, and $\frac{1}{4}(u - \frac{1}{2})/(\frac{1}{4} + 1/n)$ for $k = 3$, as unit normal deviates, where k is the number of means in the group. These approximate formulas are recommended for routine work since they are more easily handled than the exact tables given by K. R. Nair [Biometrika **35**, 118–144 (1948); these Rev. **9**, 602]. The author makes the following statement about the significance level for the whole test procedure. "It is hard to see how to find the frequency of false positives with the whole system analytically, but the writer conjectures that, if the same level, such as 5%, is used in all three tests, the frequency of false positives will be between 1.2 and 1.6 times the level used (i.e., between 6% and 8% when a 5% level is used). This is about where the frequency of false positives stands for many repeated and result-guided tests of significance now in actual practice."

D. M. Sandelius (Seattle, Wash.).

TOPOLOGY

Heawood, P. J. Map-colour theorem. Proc. London Math. Soc. (2) **51**, 161–175 (1949).

Further study of an arithmetic problem equivalent to the four color theorem when restrictions on certain constants make them correspond to a map. The author attempts to show that the arithmetic problem is solvable even without the geometric restrictions. He does this for a large number of cases, but does not claim completeness. Thus his efforts afford only a method by which the four color problem might be solved if all cases could be disposed of.

P. Franklin (Cambridge, Mass.).

Areškin, G. Ya. Compacta and their structure. Mat. Sbornik N.S. **25(67)**, 151–154 (1949). (Russian)

Let X be a compact metric space, and let $\{U\}$ be any countable open basis for X such that $0, X \in \{U\}$, and such that $\{U\}$ is closed under the formation of finite unions and finite intersections. The family of sets $\{U'\}$ is then a countable distributive lattice with zero and unit elements. The author proves that a countable lattice S is lattice-isomorphic to a lattice $\{U'\}$ for some compact metric space if and only if: (1) S has zero and unit elements; (2) S is distributive; (3) if $0 \neq b < a$, then there exists a c such that $0 \neq c < a$ and

$bc=0$; (4) if $a\neq 0\neq b$ and $ab=0$, then there exist a_1 and b_1 such that $a\leqq a_1$, $b\leqq b_1$, $a_1b=0=b_1a$, and $a_1+b_1=1$. Necessity is obvious, and the sufficiency is proved by the usual method of topologizing the space of maximal ideals in S.

E. Hewitt (Seattle, Wash.).

Sikorski, R. On the separability of topological spaces. Colloquium Math. 1, 279–284 (1948).

The author considers six properties of a topological space X, ranging from (1) X has a countable base to (6) every collection of disjoint open sets of X is countable. Completing results of Marczewski [Fund. Math. 34, 127–143 (1947); these Rev. 9, 98], the author shows by examples that, apart from the known implications between these properties, no other logical connections can hold among them for all topological spaces. *A. H. Stone.*

Sikorski, R. Remarks on a problem of Banach. Colloquium Math. 1, 285–288 (1948).

The problem in question is to characterize those metric spaces X which have the property: (B) there exists a continuous one-to-one mapping of X onto a compact metric space. After giving simple examples of spaces with and without this property, the author gives a characterization in terms of semi-continuous decompositions, and deduces that the complement, in a compact metric space, of a null sequence of disjoint closed sets has property (B). [A more general theorem is stated (theorem (ii), p. 287) but the argument given is inadequate, as the decomposition F there constructed will not in general be semi-continuous.]

A. H. Stone (Manchester).

Kuratowski, C. Une méthode de prolongement des ensembles relativement fermés ou ouverts. Colloquium Math. 1, 273–278 (1948).

Let E be a given nonempty subset of a metric space 1; for each $X\subset E$, define

$$F(X)=\{x\,|\,\rho(x, X)\leqq\rho(x, E-X)\},$$
$$G(X)=\{x\,|\,\rho(x, X)<\rho(x, E-X)\}$$

(with the convention $\rho(x, 0)=\infty$); thus $G(X)=1-F(E-X)$. The author derives a number of properties of F and G, notably $F(X\cup Y)=F(X)\cup F(Y)$ and $E\cap F(X)=E\cap\bar{X}$, with dual properties for G. It is deduced very simply that if $\{X_\lambda\}$ is an arbitrary family of subsets of E, open relative to E, there exist open subsets G_λ of 1 such that (i) $X_\lambda=E\cap G_\lambda$, (ii) whenever the intersection of a finite number of sets $X_{\lambda_1}, \cdots, X_{\lambda_n}$ is empty, then $G_{\lambda_1}\cap\cdots\cap G_{\lambda_n}=0$ and

$$\bar{G}_{\lambda_1}\cap\cdots\cap\bar{G}_{\lambda_n}=\bar{X}_{\lambda_1}\cap\cdots\cap\bar{X}_{\lambda_n}.$$

A dual theorem applies to relatively closed subsets of E.

A. H. Stone (Manchester).

Ramanathan, A. On the strong extension of a T_1-space into a T_1-bicompact space. J. Indian Math. Soc. (N.S.) 13, 25–30 (1949).

A bicompact topological space R is said to be minimal-bicompact if any strengthening of the topology of R destroys the property of bicompactness. The author observes that a T_1-space R can be imbedded in a bicompact space $R\cup\{\gamma\}$ in a variety of ways, and that $R\cup\{\gamma\}$ is minimal-bicompact if and only if the imbedding is the canonical one. Various results are proved concerning the local regularity of $R\cup\{\gamma\}$, and an example is exhibited of an irregular (and thus non-Hausdorff) minimal bicompact space.

E. Hewitt (Seattle, Wash.).

Nagao, Hirosi. On the topologies of homeomorphism groups of topological spaces. Osaka Math. J. 1, 43–48 (1949).

The author investigates the g-topology for homeomorphism groups [reviewer's terminology, Amer. J. Math. 68, 593–610 (1946); these Rev. 8, 479]. The results obtained, presumably independently, are essentially included in the paper cited above. *R. Arens* (Los Angeles, Calif.).

Cohen, L. W., and Goffman, Casper. A theory of transfinite convergence. Trans. Amer. Math. Soc. 66, 65–74 (1949).

The theory of complete uniform spaces, while embodying generalizations of many classical properties of complete metric spaces, has not previously produced a generalization of the theory of category. The present paper develops such a generalization for a special class of uniform spaces. Let ξ^* be any limit ordinal such that no single-valued image of an initial segment $E[\gamma;\gamma<\delta<\xi^*]$ is cofinal with ξ^*. The uniform spaces considered are those with entourages in $S\times S$ well-ordered by set-inclusion of type ξ^*, and satisfying also a strong axiom on inclusion of neighborhoods. For such spaces, any intersection of fewer than $|\xi^*|$ open sets is open ($|\alpha|$ = cardinal number of the ordinal number α). Completeness is defined in the manner usual for uniform spaces. A set of the first (second) category is one which is (is not) the union of $|\xi^*|$ nowhere dense sets. It is proved that every complete space of the type considered is of the second category. Spaces of the type considered are constructed for every appropriate ordinal as order-completions of power-series fields, and an example is produced to show that the category theorem fails if the strong neighborhood axiom is weakened only slightly. *E. Hewitt.*

Stone, A. H. Incidence relations in unicoherent spaces. Trans. Amer. Math. Soc. 65, 427–447 (1949).

This paper is a study of relations between the numbers of components of sets and their frontiers, unions, and intersections. The topological space S is assumed to be connected, locally connected, and unicoherent (in the sense that S is connected and that if $S=A\cup B$, where A and B are closed and connected, then $A\cap B$ is connected); normality is not required, and most of the theorems are extremely general, holding true for sets which need not be open or closed. Definitions: the sets A and B are separated if $\bar{A}\cap B=0=\bar{B}\cap A$; Fr (A) is the frontier of A; $b_0(A)$ is one less than the number of components of A; $b_0(A)$ is thus an integer or ∞, with the convention that in interpreting an inequality involving the latter case, we first transpose all negative terms; given a sequence $A_1, A_2, \cdots, A_n$ of sets, X_r $(1\leqq r\leqq n)$ is the set of all points of S which lie in A_i for r or more values of i. Typical theorems are as follows. (I) If in the definition of unicoherence we use open sets A, B instead of closed sets, an equivalent definition is obtained. (Numerous other characterizations of unicoherence are given, as well as sufficient conditions which hold in locally arcwise connected Hausdorff spaces.) (II) Given $A_1, A_2, \cdots, A_n$, such that A_i-A_j and A_j-A_i are separated for $i\neq j$, then $\sum_1^n b_0(A_i)\leqq\sum_1^n b_0(X_r)$. If Fr $(A_i)\cap$Fr $(A_j)=0$ for $i\neq j$, or if the sets A_i are closed and Fr $(A_i)\cap$Fr $(A_j)\cap$Fr $(A_i\cup A_j)=0$, then the equality holds. In particular, given A, B such that Fr $(A)\cap$Fr $(B)=0$, then $b_0(A\cap B)+b_0(A\cup B)=b_0(A)+b_0(B)$. (III) If $\bar{A}$ and $\bar{B}$ are connected and Fr $(A)\cap$Fr $(B)\neq 0$, then $b_0(\bar{A}\cap\bar{B})\leqq b_0(\text{Fr }(A)\cap\text{Fr }(B))$. (IV) Let $A_1, A_2, \cdots, A_n$ be n connected sets, every $n-1$ of which, but not all of which,

have a point in common; and suppose $n \geqq 3$. Then there are at least $2n-3$ pairs of different integers i, j, such that $\mathrm{Fr}(A_i) \bigcap \mathrm{Fr}(A_j) \neq 0$. (This is a best possible result for each n, even when S is the plane and the A_i's are closed connected sets with connected complements.)

E. E. Moise (Princeton, N. J.).

Stone, A. H. Incidence relations in multicoherent spaces. I. Trans. Amer. Math. Soc. 66, 389–406 (1949).

This paper represents a continuation of the author's study of relations between the components of unions and intersections of sets and their frontiers in connected and locally connected spaces. Omitting the assumption of unicoherence made in the paper reviewed above, generally analogous results are obtained in terms of the degree of multicoherence of the space. It is shown that, whereas some of the expected generalizations break down in the multicoherent case, valid generalizations can be established although the statements are somewhat more complicated. This is true in particular in the Phragmén-Brouwer theorem which is proved in the general setting and which is shown to reduce to the natural extension of this theorem in case the space under consideration has no local cut points. On account of the need for methods of proof depending on decomposition and approximation, it is found advantageous to add an assumption of complete normality on the spaces considered, which was not necessary in the unicoherent case.

G. T. Whyburn (Charlottesville, Va.).

Fox, Ralph H. A remarkable simple closed curve. Ann. of Math. (2) 50, 264–265 (1949).

A simple closed curve Γ in ordinary three-space R is constructed such that (1) the fundamental group of $R-\Gamma$ is not Abelian, (2) there are a point p on Γ and arbitrarily small neighborhoods V of p such that R admits a homeomorphism onto itself which leaves V pointwise fixed and maps $\Gamma - V$ onto a subset of a plane of R.

S. Eilenberg (New York, N. Y.).

Keldyš, Lyudmila. A continuous mapping of a segment on an n-dimensional cube. Doklady Akad. Nauk SSSR (N.S.) 66, 327–330 (1949). (Russian)

A continuous transformation $F(X)$ is called uniformly non-dimension-raising if $\dim F(\bar{V}) \leqq \dim \bar{V}$ for every open set V in X, and is called bi-valent if the inverse sets contain at most two points. It is shown that certain transformations (named, but not defined) of the segment I into the n-dimensional cube can be factored into the product of a finite number of transformations, alternately from each of the classes above, the bi-valent transformations increasing dimension by exactly one. The results follow by repeated application of the following stated theorems. A proof of the first of these is sketched in the note.

(I) Let $Y=f(I)$ be a continuous mapping of the segment on an n-dimensional compactum Y, satisfying the following two conditions: (1) the image $f(d)$ of an arbitrary segment d of I contains a subset which is homeomorphic to the n-dimensional cube; (2) for every pair of nonintersecting segments d_1 and d_2:

$$\dim (f(d_1) \cdot f(d_2)) \leqq n-1, \quad d_1 \cdot d_2 = 0.$$

Then the mapping f can be expressed in the form of a superposition of two continuous mappings $f=uv$, where u is bivalent, $\dim v(I) = n-1$ and the image $v(d)$ of an arbitrary segment d of I contains a subset homeomorphic to the $(n-1)$-dimensional cube.

(II) Let $Y=f(I)$ be the continuous image of a segment I of an n-dimensional compactum such that the image $f(d)$ of an arbitrary segment of I has a subset homeomorphic to an n-dimensional cube. Then f can be represented as the superposition of two continuous mappings $f=gf_1$, where g has zero-dimensional inverses and is uniformly non-dimension-raising, and f_1 satisfies properties (1) and (2) of theorem I.

L. Zippin (Flushing, N. Y.).

Sitnikov, K. The definition of the dimension of a closed set by a metric property of the complementary space. Doklady Akad. Nauk SSSR (N.S.) 66, 1059–1062 (1949). (Russian)

Let K be a compact subset of Euclidean R^n and let $W=R^n-K$. Any $(n-1)$-dimensional cycle of W linked with every point of K will be called a K-sack. Let $\alpha^p K$ denote the lower bound of all $\epsilon>0$ such that an ϵ-deformation exists carrying K into a p-dimensional set. Let z be any true cycle in R^n, and let μz denote the upper limit of all $\epsilon>0$ for which there exists a K which is a carrier of z and is such that z does not bound in the ϵ-neighborhood of K. If z bounds in every carrier, set $\mu z=0$ (z is inessential). Now if z is a cycle of W denote by $\alpha_W^p z$ the lower bound of all $\alpha^p K$ for compacta K carrying cycles homologous to z in W. For convenience set $\alpha_W^{-1} z = 1$. The principal theorem may now be stated. If $\dim K = r$, $0 \leqq r \leqq n-1$, there exists an $\alpha>0$ such that if X is a K-sack then $\alpha_W^{r-1} X > 0$, but if X is any cycle in W, $\alpha_W^r X = 0$. There always exist K-sacks X for which μX is arbitrarily small. If $\dim K = n$, there is an $\alpha>0$ such that every K-sack z^{n-1} has $\mu z^{n-1} > \alpha$ (at the same time: $\alpha_W^{n-2} z^{n-1} > 0$, and, trivially, $\alpha_W^{n-1} z^{n-1} = 0$). The author sketches a proof which contains the following principal lemma: let $\bar{Q}$ be an n-dimensional polyhedron in R^n which is the closure of its open kernel Q. Let $p<n-1$ and let z^p be a cycle which bounds in Q. Then if we denote by Q' the boundary of $\bar{Q}$, we get $\alpha^p Q' \geqq \mu z^p$.

L. Zippin.

Boltyanskiĭ, V. An example of a two-dimensional compactum whose topological square is three-dimensional. Doklady Akad. Nauk SSSR (N.S.) 67, 597–599 (1949). (Russian)

Let p be an arbitrary, but fixed, prime and let k be an arbitrary, but fixed, positive integer. The author constructs a space P (lying in four-dimensional Euclidean space), $P=P(p, k)$, such that $\dim P = 2$ but $\dim (P \times P) = 3$, as follows. Begin with an annulus K, with edges α and β. Identify as a single point each set of p points of α which divide it into p equal arcs and similarly identify each set of p^k points of β dividing β into p^k equal arcs. The resulting polyhedron is called a leaf of order k, and is denoted by Π_k. Its edges a and b are the "circles" which result from α and β respectively after the identifications. Now if a circular hole is cut out of some two-cell, and the edge a of Π_k is matched with the edge of this hole, then the hole of the two-cell is said to be overlaid with the leaf Π_k. Next one defines a tower $\Pi_{k,l}$ depending on the integers k and l, as follows. Start with a leaf Π_k. Cut a hole in it, not touching the edges of the leaf, and overlay this hole with a leaf Π_{k+1}; in that leaf cut out a hole not meeting its edges and overlay this hole with a leaf Π_{k+2}. Continue this leaf-upon-leaf to the lth stage. The resulting polyhedron is the desired tower.

Let S^2 be a triangulated two-dimensional sphere. In each simplex, cut out a hole and overlay with the leaf Π_k. Let $P_{k,k}$ denote the resulting space, and suppose now that somehow there is constructed a triangulated surface $P_{k,l}$

($l \geqq k$) consisting of a "riddled" sphere and overlaid leaves of respective orders $k, k+1, \cdots, l$. Now, cut a hole in each simplex of $P_{k,l}$. The holes belonging to S^3 are to be overlaid with $(k, l+1)$-towers, but the holes belonging to a leaf Π_i are overlaid with an $(i+1, l+1)$-tower. This gives the polyhedron of the next stage, denoted by $P_{k,l+1}$. The constructions can be thought of as carried out in a four-space, the simplexes of successive triangulated surfaces approaching zero in diameter. The resulting surfaces $P_{k,n}$ converge with increasing n to a limiting surface denoted by P_k. This is the desired space P. The proof is sketched in some detail.

L. Zippin (Flushing, N. Y.).

Lunc, A. A bicompactum whose inductive dimension is greater than its dimension defined by means of coverings. Doklady Akad. Nauk SSSR (N.S.) **66**, 801–803 (1949). (Russian)

Let ω be the first transfinite ordinal corresponding to the power of the continuum, and let T be a well-ordered set of "points" of ordinal character $\omega+1$. Let S be the linear continuum (non-Archimedean) obtained by the standard procedure of joining every pair of consecutive points, α and $\alpha+1$, of T by an interval I_α: $0 \leqq a_\alpha \leqq 1$. Let R_3 denote the topological product of S and I ($0 \leqq t \leqq 1$). The author constructs in R_3 a closed nondense subset R such that (1) R contains a point x and a neighborhood V of x such that every neighborhood U of x with $U \subset V$ has a one-dimensional boundary and, on the other hand, (2) every finite covering of R by open subsets has a refinement of order two. Consequently, (1) ind $R=2$ and (2) dim $R=1$.

The following set F is vital to the construction of R. Let Q denote the unit square: $0 \leqq x, y \leqq 1$. Let C be a Cantor discontinuum on the x-axis and let K consist of the unit segments which project into C. Let L_{ik} denote a segment in Q projecting into an interval complementary to C, parallel to the x-axis at the ordinate $y=k/(i+1)$, $k=1, \cdots, i$; $i=1, \cdots, n, \cdots$. The set F is the sum of K and all intervals L_{ik}. To each α of the set T there is constructed a set F_α which is homeomorphic to a section of F obtained by displacing F in Q parallel to the x-axis. This displacement, which will not be described here, depends on a well-ordering of the points of the square. The set F_α is defined in $Q_\alpha = I_\alpha \times I$. The set R is the sum of all the F_α and the set $T \times I$. Let s denote the "last" point of T. The point x, above, is any inner point of $s \times I$.

L. Zippin.

Lokucievskiĭ, O. V. On the dimension of bicompacta. Doklady Akad. Nauk SSSR (N.S.) **67**, 217–219 (1949). (Russian)

The author constructs a bicompactum S which is the sum of two closed subsets S_1 and S_2, each of inductive dimension equal to one: ind S_1 = ind S_2 = 1, but such that ind $S=2$. This gives another, and simpler, example of a space for which the dimension as defined by coverings is less than the inductive one [compare with the paper reviewed above]. Let P denote the standard transfinite bicompact "interval," and w_1 its last point [cf. Lunc, loc. cit.], let C denote the Cantor discontinuum, and let R be the topological product of P and C: $R=P \times C$. If one identifies the end-points of intervals of C, one gets the linear segment $0 \leqq t \leqq 1$. Do this to the subset of R given by $w_1 \times C$, denote the resulting interval by F, and let M denote the countable dense set which images the set of endpoints of intervals. Let R' denote the image of R. Prepare two copies R_1' and R_2' of R', with F_1 and F_2, M_1 and M_2 as the associated subsets. Select in F_2 a subset N homeomorphic to M_2 but having no points in common with M_2 (other than the extreme endpoints "0" and "1" of the interval F_2). Let $g(F_1)=F_2$ be a homeomorphism which carries N into M_1. Let Q denote the sum of the spaces R_1' and R_2', and let S denote the decomposition space of Q, where F_1 and F_2 are identified as determined by g. This S is the desired space. The author also establishes the results: dim R = dim S, and ind $R \leqq$ ind $S+1$ (when these are finite), where R and S are bicompacta of the same power, R being a dyadic enveloping space to S. [See P. Alexsandrov, Uspehi Matem. Nauk (N.S.) **2**, no. 1(17), 5–57 (1947); these Rev. **10**, 389.]

L. Zippin (Flushing, N. Y.).

Vaccaro, Michelangelo. Automorfismi e struttura topologica di un certo complesso regolare. Comment. Math. Helv. **23**, 18–25 (1949).

Considérons un polygone P de $n \geqq 2k+1$ sommets numérotés, et les $k-1$ cellules dont les sommets sont obtenus en prenant dans P les sommets 1, $k+1$, et tous les sommets intermédiaires sauf un quelconque d'entre eux. Si on applique à ces cellules le groupe des substitutions circulaires des sommets de P on obtient un complexe C. Pour $k \geqq 3$ les groupes d'automorphismes de C et de P sont isomorphes. Le complexe C est une variété, qui peut être, soit le produit topologique d'une hypersphère avec une circonférence, soit le produit topologique d'une hypersphère avec un segment, les hypersphères terminales étant alors soudées de façon que la variété obtenue soit non orientable. La non-orientabilité se produit pour k pair, n impair. L'auteur termine en étudiant le passage d'un complexe C à un complexe du même type ayant un sommet de moins.

L. Gauthier.

Vaccaro, Michelangelo. Sopra una generalizzazione del processo di dualizzazione di Poincaré per un complesso topologico. Ann. Mat. Pura Appl. (4) **27**, 107–113 (1948).

The author describes a scheme which interpolates between a (combinatorial) r-dimensional manifold K and its dual K^* a series of $r-2$ complexes $K_2, \cdots, K_{r-1}$, such that $K^*=K_1, K_2, \cdots, K_{r-1}, K_r=K$ all have a common barycentric subdivision K'. An r-cell of K_p is obtained by taking together all r-simplices of K' which have in common the barycenter of a p-simplex of K; the other cells are obtained in a similar fashion. Actually the scheme is formulated for a more general type of complex.

H. Samelson.

*Cartan, Henri. **Algebraic Topology.** Lectures edited by George Springer and Henry Pollak. Published by the editors, 223 Pierce Hall, Harvard University, Cambridge 38, Mass. vi+169 pp. $2.00 plus postage.

The work consists of notes based on a course given at Harvard University in 1948. The essential feature is a comprehensive homology (or cohomology) theory of locally compact topological spaces, with the notion of grating playing the central role. Although a notion due to Alexander and carrying the same name is included as a particular case, the present treatment is characteristically axiomatic. After four chapters dealing with the basic concepts of algebraic topology and exterior differential forms, the author proceeds to consider graded rings A with a differential operator d such that $d^2=0$. With respect to d the derived ring is defined. This notion will be given a topological significance when the supports of elements in A are defined. The latter are functions in A, with values which are closed subsets of a locally compact topological space, satisfying

certain conditions. A grating is a graded ring with differential operator and supports. It is shown that this theory includes the de Rham theory of differential forms on a differentiable manifold, the Alexander grating theory, the Čech and singular homology theories. The coefficient ring $\mathfrak{A}$ comes into play by considering it as a grating with every element having degree zero, with vanishing differential operator, and with supports defined to be either the whole space or empty. Call A an $\mathfrak{A}$-grating if $\mathfrak{A}$ can be identified as a subgrating of A. If, moreover, $\mathfrak{A}$ is isomorphic to the derived ring $H(A/A_M)$ at every point M of E, A is called $\mathfrak{A}$-simple. The culminating result of this work is an isomorphism theorem which characterizes the cohomology ring of a compact space and may be stated as follows. Let E be a compact space, A, B, $\mathfrak{A}$-grating and $\mathfrak{B}$-grating, respectively, which are fine and are $\mathfrak{A}$-simple and $\mathfrak{B}$-simple, respectively. If there is an isomorphism f of $\mathfrak{A}$ onto $\mathfrak{B}$, the induced homomorphism $\hat{f}$ of $H(A)$ into $H(B)$ is an isomorphism onto. *S. Chern* (Chicago, Ill.).

Fet, A. The homology ring of the space of closed rectifiable curves on a sphere. Doklady Akad. Nauk SSSR (N.S.) 66, 347–350 (1949). (Russian)

The Betti groups, coefficients modulo two, of the space Ω of closed rectifiable curves on the two-dimensional sphere were analyzed by Morse [The Calculus of Variations in the Large, Amer. Math. Soc. Colloquium Publ., v. 18, New York, 1934]. This note constructs the corresponding homology ring of Ω (coefficients modulo two). The first theorem gives the appropriate multiplication table, expressed in terms of ∇-cycles. Theorem 2 establishes certain normal forms for the fundamental cycles defined on Ω.

L. Zippin (Flushing, N. Y.).

Fet, A. Integral homology of the space of closed curves on a sphere. Doklady Akad. Nauk SSSR (N.S.) 66, 569–570 (1949). (Russian)

This note analyzes the homology groups with integer coefficients of the "function space" Ω of closed rectifiable curves on a two-dimensional sphere. The result is the theorem that $\Delta_{2k}(\Omega)=0$, $\Delta_{2k+1}(\Omega)=\Pi_0+\Pi_2$, where Π_0 is a free cyclic group and Π_2 is cyclic of order two. The author remarks that this gives the first example of torsion coefficients in a function space. *L. Zippin* (Flushing, N. Y.).

***Brändli, Emil Rudolf. Beiträge zur Theorie des Cohomologieringes.** Thesis, Eidgenössische Technische Hochschule, Zürich, 1948. 75 pp.

In part I, the problem of constructing a polytope with cohomology ring isomorphic to a given ring R is solved in case R has "dimension" not exceeding 2. The main result is as follows. Let an arbitrary ring R with unit be given which (a) is a hypercomplex system over the rationals, (b) has a finite basis, of which p_2 are of dimension 2, p_1 of dimension 1, and the unit only of dimension zero, (c) the multiplication of the one-dimensional elements is anticommutative. Form the multiplication table $(\xi_i\xi_j)=\sum_\nu a_{(ij)}^\nu\eta_\nu$ $(i<j)$ composed of $N=\binom{p_1}{2}$ rows and p_2 columns. If $\|a_{(ij)}^\nu\|$ has rank r, then a new 2-basis can be chosen so that $(\xi_i\xi_j)_k=\eta_k$, $k=1,2,\cdots,r$; $(\xi_i\xi_j)_k=\sum_{i=1}^{r}b_{ki}\eta_i$, $k=r+1,\cdots,N$, and the 2-polytope with co-ring isomorphic to R can be now described: it consists of r closed orientable surfaces F^j of genus q^j (the number of entries in the jth column) on which certain identifications are made, and p_2-r spheres S_2.

Part II is more original and discusses the problem of the position of a complex in S_n and E_n [Alexandroff and Hopf, Topologie I, Springer, Berlin, 1935, p. 449]. Let P be a subpolytope of a sphere given in a definite simplicial subdivision. The vertices of a barycentric subdivision of S_n that do not lie in P generate a polytope Π. It is shown that, given polytopes X, Y in S_n, the co-ring $R(\Pi)$ of the Π corresponding to $X\bigcup Y$ depends not only on the homology structures of X and Y, but also on a certain system of "linking cycles." Isomorphism of the $R(\Pi)$ is necessary, and in some cases, but not in general, sufficient, for equal position; the usefulness of the $R(\Pi)$ is shown by an example of homeomorphic polytopes with nonisomorphic $R(\Pi)$'s. A definition of the cup-product stressing its connection with the Cartesian product is given in part III. Let K be a finite polytope, and identify K with the diagonal in $K\times K$: $\sigma\leftrightarrow{}^+\sigma$: $\{(p,p)\,|\,p\in\bar{\sigma}\}$. For every cycle $z_p\subset K$, ${}^+z_p$ is a cycle in a simplicial subdivision of $K\times K$. Taking proper coefficient domains, and letting $Z(\ )$ denote the class of the object in parentheses, a product $Z(\xi^p)\circ Z(\eta^q)\in H^{p+q}(K)$ is defined by the condition that

$$KI[Z(\xi^p)\circ Z(\eta^q),\ Z(z_{p+q})]=KI[Z(\xi^p\times\eta^q),\ Z({}^+z_{p+q})]$$

for every homology class $Z(z_{p+q})$, and it is shown that $Z(\xi^p)\circ Z(\eta^q)=Z(\xi^p)\bigcup Z(\eta^q)$. *J. Dugundji.*

Blakers, A. L., and Massey, William S. The homotopy groups of a triad. Proc. Nat. Acad. Sci. U. S. A. 35, 322–328 (1949).

A triad $(X;A,B)$ is an arcwise connected space X with subsets A, B such that $A\bigcap B\neq 0$. By E^n denote a solid n-sphere, by E_+^{n-1} (E_-^{n-1}) the surface of its upper (lower) hemisphere, $p_0\in E_+^{n-1}\bigcap E_-^{n-1}=S^{n-2}$ and $x_0\in A\bigcap B$. In the functional space of all f: $(E^n, E_+^{n-1}, E_-^{n-1}, p_0)\to(X;A,B,x_0)$ the arcwise connected components form a group $\pi_n(X;A,B,x_0)$ for $n\geqq 3$, which is Abelian for $n>3$ and has $\pi_1(A\bigcap B,x_0)$ as a group of operators. Several of the properties of the triad groups are as follows. (1) Using homomorphisms ∂: $\pi_n(X;A,B,x_0)\to\pi_{n-1}(A,A\bigcap B,x_0)$ induced by restricting the maps to E_+^{n-1}, and those induced by the identity maps i: $(A\ A\bigcap B)\to(X,B)$, j: $(X,x_0,B)\to(X;A,B)$, the $(X;A,B)$-sequence

$$\cdots\to\pi_n(A,A\bigcap B)\xrightarrow{i}\pi_n(X,B)\xrightarrow{j}\pi_n(X;A,B)\xrightarrow{\partial}\pi_{n-1}(A,A\bigcap B)\to\cdots$$

is exact. (2) If X^+ is the space X with a cell $\mathcal{E}^n$ adjoined, and if $\pi_i(\mathcal{E}^n\bigcap X)=0$ for $i\leqq m$, then

$$\partial:\pi_{i+1}(X^+;X,\mathcal{E}^n)\to\pi_i(\mathcal{E}^n,\mathcal{E}^n\bigcap X)$$

is a zero homomorphism for $2\leqq i\leqq m+n-1$. (3) If $H^n(X,A\bigcup B)=H^{n+1}(X,A\bigcup B)=0$ (integral relative cohomology groups), then there is a homomorphism of $\pi_n(X;A,B)\to G_n$, where G_n is the group of bilinear maps $[H^p(X,A),\ H^q(X,B)]\to H^n(E^n,S^{n-1})$, $p+q=n+1$; for $(X;A,B)=(S^n;E_+^n,E_-^n)$, G_n is infinite cyclic, and h is a homomorphism on. (4) If $A\vee B=(A\times b_0)\bigcup(a_0\times B)\subset A\times B$, then for $n\geqq 3$ $\pi_n(A\vee B,A,B)\approx\pi_{n+1}(A\times B,A\vee B)$, and in the $(A\vee B,A,B)$-sequence, the ∂ homomorphism is trivial; in addition, $\pi_j(S^p\vee S^q;S^p,S^q)\approx\pi_j(S^{p+q-1})$ for

$$j\leqq p+q+\min(p,q)-3.$$

(5) If $\varphi_n(S^n;E_+^n,E_-^n)\to(S_1^n\vee S_2^n,S_1^n,S_2^n)$ identifies all points on the equator $S^{n-1}\subset S^n$ to a single point, then there is defined a homomorphism $\varphi_n j$: $\pi_r(S^n)\to\pi_r(S^{2n-1})$ for $r\leqq 3n-3$. The homomorphism $\varphi_n j$ coincides with the generalized Hopf homomorphism originally defined by G. W. Whitehead [same Proc. 32, 188–190 (1946); these Rev. 8, 50] only for

$r \leqq 3n-4$. This extension of the range of the homomorphism accomplished by means of triads, an application of the techniques of Whitehead yields the first information about $\pi_6(S_3)$: it is nontrivial.

The importance of this new group lies essentially in the following considerations: if $X = A \cup B$, the "excision" map $i: (A, A \cap B) \to (A \cup B, B)$ induces isomorphisms of the relative homology groups, but not, in general, of the relative homotopy groups (this is the basic difference between these two systems of groups); the groups $\pi_n(X; A, B)$ now available measure the deviation of the excision map from isomorphism. Use of the triad sequence also permits generalization and unification of existing results, based on a systematic use of obstruction theory. Thus, in the $(S^n; E_+^n, E_-^n)$-sequence, the excision $i: \pi_p(E_+^n, S^{n-1}) \to \pi_p(S^n, E_-^n)$ is essentially the Freudenthal "suspension" $\pi_{p-1}(S^{n-1}) \to \pi_p(S^n)$, and (2), (3) contain all the Freudenthal theorems, with several improvements. *J. Dugundji* (Los Angeles, Calif.).

Whitehead, J. H. C. The homotopy type of a special kind of polyhedron. Ann. Soc. Polon. Math. 21 (1948), 176–186 (1949).

The results of a previous paper [J. H. C. Whitehead, Comment. Math. Helv. 22, 48–92 (1949); these Rev. 10, 559] are extended to the case where K is a finite polytope, $\dim K \leqq n+2$, and $\pi_i(K) = 0$ for all $i < n$ (an A_n^2 polytope; the case $n = 2$ was treated in the above paper). The problem appears to be simpler when $n > 2$. The definition of an "A_n^2-ring" ($n > 2$) differs from that of an "A_2^2-ring" in just two respects: only the cohomology groups over the integers and over the integers mod 2 are included, and the Pontrjagin square is replaced by the Steenrod square assigning to each s-cocycle mod 2 an $(s+2)$-cocycle mod 2; the various types of homomorphisms required in the A_2^2 case remain, but are altered accordingly, as is also the notion of a "proper" homomorphism. It is then shown that two A_n^2-polytopes ($n > 2$) belong to the same homotopy type if and only if their "A_n^2-rings" are properly isomorphic. The proof follows the same pattern as the argument in the A_2^2 case: any abstractly given "A_n^2-ring" is realizable as the "A_n^2-ring" of some A_n^2 polytope, and a homomorphism of one "A_n^2-ring" into another is proper if and only if it is induced by a continuous mapping of their realization polytopes. It is also shown that if K is an A_n^2-polytope, and $\pi_{n+1}(K) = 0$, then the homotopy type of K is determined solely by its Betti-groups and torsions, and a normal form for such polytopes and their "rings" is given. *J. Dugundji.*

Whitehead, J. H. C. Combinatorial homotopy. I. Bull. Amer. Math. Soc. 55, 213–245 (1949).

The author's purpose is to establish an algebraic apparatus which does for the homotopy type and n-homotopy type [cf. below] what groups do for the 2-type (two complexes have the same 2-type if their fundamental groups are isomorphic). The author has achieved this previously for 3-dimensional complexes [Ann. of Math. (2) 42, 1197–1239 (1941); these Rev. 3, 142] and for finite, simply connected 4-dimensional complexes [Comment. Math. Helv. 22, 48–92 (1949); these Rev. 10, 559]. In this paper he defines the CW-complex, a space which is union of cells e_i^n; the $\overline{e_i^n}$ are images of n-simplices under maps which are homeomorphic in the interior, and which send the boundary into the $(n-1)$-skeleton of the space; C means closure finite, and W is weak topology, i.e., a set is closed if its intersection with every $\overline{e_i^n}$ is closed. Many topological properties of CW-complexes are developed, which bring them close to ordinary polyhedra, e.g., local contractability. The n-type (formerly called n-group) is defined: two maps of f, g of X into Y are n-homotopic, $\simeq_n$, if, for every map φ of any CW-complex of dimension not exceeding n into X, $f\varphi$ and $g\varphi$ are homotopic. If X is a CW-complex, this means $f|X^n \simeq g|X^n$ ($X^n = n$-skeleton). Call X and Y n-equivalent, $=_n$, if there exist $f: X \to Y$ and $f': Y \to X$, such that $ff' \simeq_n 1$, $f'f \simeq_n 1$. (For $n = \infty$ this is defined to mean homotopy type). Two CW-complexes K, L have the same n-type if $K^n =_{n-1} L^n$. For the class α of spaces "dominated" by CW-complexes, theorems of the following type are proved: $f: X \to Y$ is an $(N-1)$-equivalence if the induced maps f_n of the homotopy groups are isomorphic onto for $n < N$. (Here N is the maximum of the (pseudo-)dimension of X and Y.) A J_m-complex K is a CW-complex for which the natural map of the $\pi_n(K^n)$ into the $\pi_n(K^n, K^{n-1})$ is an isomorphism for $n = 1, \cdots, m$. The J_m-property is an invariant of the m-type. For a simply connected J_m-complex it is proved (generalized Hurewicz theorem) that $\pi_n(K) \approx H_n(K)$ for $n = 1, \cdots, m$, and that $H_{m+1}(K)$ is spherical. Finally a general theorem on invariance of homotopy type under shrinking of a subset is proved. *H. Samelson* (Ann Arbor, Mich.).

Whitehead, J. H. C. Combinatorial homotopy. II. Bull. Amer. Math. Soc. 55, 453–496 (1949).

In this paper the fundamental concept is the homotopy system, which is a system of groups ρ_n and an operator $d: \rho_n \to \rho_{n-1}$ with properties and relations abstracted from those of the relative groups $\pi_n(K^n, K^{n-1})$ of a complex K. Algebraic lemmas, concerning in particular the system formed by ρ_1 and ρ_2, termed a free crossed module, are established. A homotopy system has a geometric realization (as $\pi_n(K^n, K^{n-1})$ for some K) if (a) $\rho_n = 0$ for $n > 4$ or (b) $\bar{\rho}_1 = \rho_1/d\rho_2$ is trivial. Homomorphisms, equivalence, etc., of homotopy systems are defined algebraically (guided by the geometric meaning). The relations between maps of one complex into another and algebraic maps of the homotopy systems of the complexes are discussed. $\varphi: K \to L$ is a homotopy equivalence if the induced homomorphism of the homotopy systems is an (algebraic) equivalence. If $\varphi, \varphi': K \to L$, L a J_m-complex [see the preceding review], $\dim K \leqq m$, and the induced maps of the homotopy-system are algebraically equivalent, then φ and φ' are homotopic.

With the homotopy system is related a system of "chain-groups" (which geometrically corresponds to the chain groups of the universal covering, with the fundamental group operating). Chain mappings, equivalence, etc., are defined. For many homotopy properties the chain groups can replace the homotopy system, through a series of equivalence theorems. E.g., a map $\varphi: K \to L$ is a homotopy equivalence if the induced chain map is a chain equivalence. Examples are discussed: a chain-equivalence for lens spaces; a 5-dimensional homotopy system which has no geometric realization. Some remarks on the groups $\pi_n(K^n)$ conclude the paper. *H. Samelson* (Ann Arbor, Mich.).

Hirsch, Guy. Sur le troisième groupe d'homotopie des polyèdres simplement connexes. C. R. Acad. Sci. Paris 228, 1920–1922 (1949).

Let M be an arbitrary polytope with $\pi_1(M) = 0$, with second integral homology group $H_2(M)$ of finite rank p_2 and with no two-dimensional torsion. The author announces the construction of a fiber space $\bar{M}$ having M as base space, with fibers a Cartesian product of p_2 circumferences (each homologous to zero in $\bar{M}$) and with $\pi_1(\bar{M}) = \pi_2(\bar{M}) = 0$.

Using this result, and theorems of Gysin on homologies in fiber spaces [Comment. Math. Helv. 14, 61–122 (1941); these Rev. 3, 317], he obtains (a) if K is the kernel of the natural map $\pi_3(M)\to H_3(M)$, then $\pi_3(M)/K\approx H_3(M)$, (b) if dim $M=2$, $\pi_3(M)$ is a free group of rank $p_2(p_2+1)/2$, and (c) if dim $M=3$, and rank $H_3(M)=p_3$, then $\pi_3(M)$ is a free group of rank $p_3+p_2(p_2+1)/2$. While the author's method may achieve some unification in approach, (b) and (c) are deducible from results of J. H. C. Whitehead [Comment. Math. Helv. 22, 48–92 (1949); these Rev. 10, 559], and he does not observe that (a) is a special case of a more general theorem of Hopf [Comment. Math. Helv. 17, 307–326 (1945); these Rev. 7, 36]. *J. Dugundji.*

Hu, Sze-tsen. Structure of the homotopy groups of mapping spaces. Amer. J. Math. 71, 574–586 (1949).

Let Y be an arcwise connected topological space, X a connected finite polytope, and X_0 a subpolytope of X. In the functional space of all $f:(X, X_0)\to(Y, y_0)$, with the co-o topology, let Ω be the arcwise connected component of the constant map $O(X)=y_0$. A homomorphism $\tau_r:\pi_r(\Omega)\to\pi_r(Y)$ can be defined whose kernel K has a composition series $K=K_0\supset K_1\supset\cdots\supset K_m=1$ such that K_{n-1}/K_n ($n=1, 2, \cdots, m$) is isomorphic with the factor group of two subgroups of the cohomology group $H^n(X-X_0, \pi_{n+r}(Y))$. If $X_0\neq 0$, then $\tau_r(\Omega)=K$. If $X_0=0$, then τ_r maps a subgroup T isomorphically onto $\pi_r(Y)$ and $\pi_r(\Omega)\approx K\oplus T$ for $r>1$, while $\pi_1(\Omega)\approx K\oplus T$ if and only if T is a normal subgroup of $\pi_1(\Omega)$. The technique used is similar to that of another article of the author [Ann. of Math. (2) 50, 158–173 (1949); these Rev. 10, 393] with the result of Fox [Bull. Amer. Math. Soc. 51, 429–432 (1945); these Rev. 6, 278] that the mappings $f:E^r\to\Omega$ satisfying $f(\partial E^r)=0$ are in a one-to-one correspondence with those $F:X\times E^r\to Y$ such that $F(X\times\partial E^r\cup X_0\times E^r)=y_0$ playing the transition role.

J. Dugundji (Los Angeles, Calif.).

GEOMETRY

van Kol, J. W. A. On regular polyhedra and a theorem of Cauchy on convex polyhedra. Nieuw Tijdschr. Wiskunde 36, 237–244 (1949). (Dutch)

Schoute [Mehrdimensionale Geometrie, v. 2, Göschen, Leipzig, 1905, pp. 153–154] invited criticism of the following definition: "A polyhedron is regular if it has congruent regular faces, the same number at every vertex." This is evidently incomplete, as it would admit an irregular nonconvex icosahedron. The author remarks that the definition can be validated by inserting the word "convex," in view of the following theorem of Cauchy [J. École Polytech. (1) 9, cahier 16, 87–98 (1813), in particular, p. 96]: a convex polyhedron is determined when its faces and their arrangement are given; in other words, two convex polyhedra, having respectively congruent faces, similarly arranged, necessarily have the same dihedral angles (and thus are either directly congruent or enantiomorphous). The author points out a flaw in the classical proofs of this theorem, and supplies a remedy. *H. S. M. Coxeter* (Toronto, Ont.).

Terasaka, Hidetaka. Eine topologische Charakterisierung der kongruenten Transformationen in E^n. Osaka Math. J. 1, 1–35 (1949).

Die Gruppe der kongruenten Transformationen des n-dimensionalen Euclidischen Raumes E^n wird axiomatisch aus dem Begriff der als Axialsymmetrisierungen bezeichneten involutorischen Transformationen charakterisiert, bei denen die Punkte einer abgeschlossenen Menge des E^n, die sogenannte Achse, festbleiben, die topologisches Bild der gewöhnlichen Geraden ist. Es sei G die durch folgende Axiome charakterisierte Gruppe von topologischen Abbildungen des E^n auf sich: (1) G wird von Axialsymmetrisierungen erzeugt. (2) Es soll genau eine Axialsymmetrisierung aus G geben, die ein gegebenes Paar verschiedener Punkte festlässt. (3) Sind zwei gegen verschiedene Grenzpunkte a, b convergente Punktfolgen $\{a_n\}$ und $\{b_n\}$ gegeben und ist S_n bzw. S die das Paar a_n, b_n bzw. a, b festlassende Axialsymmetrisierung, so convergiert für jeden Punkt x die Folge der Bildpunkte $S_n(x)$ gegen $S(x)$. (4) Convergieren $\{a_n\}$ und $\{b_n\}$ gegen den selben Grenzpunkt c und convergiert $S_n(a_n)$ gegen c', so convergiert auch $S_n(b_n)$ gegen c'. Daraus lassen sich u.a. die Begriffe Senkrechtstehen, Mitte eines Segmentes, Parallelverschiebung und Umlegung einer Achse, Coordinaten auf einer Achse, Winkelhalbierende und Winkelvergleichung einführen. Erweitert man die Gruppe G durch Hinzunahme aller Centralsymmetrisierungen mit beliebigen Punkten des E^n als Centren zur Gruppe G^*, so gibt es genau eine Transformation aus G bzw. G^*, die m aufeinander senkrechte Strahlen durch einen Punkt in m aufeinander senkrechte Strahlen in einem anderen Punkt überführt. *R. Moufang* (Frankfurt am Main).

Stauder, M. Francis Borgia. Studies on projective generalizations of metric geometry. Rep. Math. Colloquium (2) 8, 49–57 (1948).

This Notre Dame thesis begins with a summary of notions and propositions due to Menger [same Rep. (2) 5–6, 60–75 (1944); these Rev. 6, 98]. The following abstract is taken from the introduction. In part I we study condition Q: for every five elements p, q, r, s, t, $(p, r, q, s)\cdot(p, s, q, t)=(p, r, q, t)$, in connection with the postulates for an F-projective space. We further investigate quintuples satisfying condition Q but containing a quadruple of pairwise distinct elements with a cross-ratio 0 and prove the existence of many-to-one projective mappings of such quintuples in the F-straight $P_1(F)$. We finally exhibit sextuples satisfying Q but containing a proper quadruple with a cross-ratio 0 for which there do not even exist many-to-one projective mappings in the F-straight $P_1(F)$. In part II we associate with each ordered quadruple of elements of a set a "cross-reciprocal," that is, an unordered pair $(f, 1/f)$ of elements of F_m and study the resulting situation which is analogous to that in Menger's $|F|$-projective spaces in which with each ordered quadruple of elements a "cross-value," that is, an unordered pair $(f, -f)$ is associated.

In part III we generalize the concept of cross-ratio of lines in terms of what we call quint-ratios of points. If L_1, L_2, L_3, L_4 is an ordered quadruple of concurrent lines of the projective plane no three of which are identical, then a number (L_1, L_2, L_3, L_4) is associated with this quadruple. If p is the common point of the four lines, and p_i a point of L_i, we write $(L_1, L_2, L_3, L_4)=(p; p_1, p_2, p_3, p_4)$. We try to axiomatize the properties of these quint-ratios. We call a set with quint-ratios a general projective line plane and characterize the subsets of the ordinary projective plane among these general planes. The result is analogous to Menger's theorem about the congruence order 5 of the Euclidean plane which states that a metric space of which

every five points are congruent with five points of the Euclidean plane is itself congruent with a subset of the Euclidean plane. In our theorem preservation of the quintratios will replace congruency (i.e., preservation of the distances), and the decisive number will be 9 instead of 5.

C. Y. Pauc (Cape Town).

DeBaggis, Henry F. Hyperbolic geometry. II. A theory of parallelism. Rep. Math. Colloquium (2) **8**, 68–80 (1948).

[This is the second part of a Notre Dame thesis, the first part of which was published in the same Rep. (2) **7**, 3–14 (1946); these Rev. **7**, 473.] Inspired by Menger the author established in part I a system of six postulates for the hyperbolic plane in terms of the operations "joining" and "intersecting" and derived the entire classical theory for linear and planar order (including the law of Pasch). The present work is devoted to a theory of parallelism under the same assumptions at first and based on the following definition: if P and Q are distinct points the line l' is called parallel to the ray P/Q if the lines $l=P+Q$ and l' do not intersect and if through Q there is a line m which intersects l' and has the following property: each line through P which does not intersect l' meets m in a point R such that through R there is at most one line which meets neither l nor l'. The line l' is parallel to the line $P+Q$ if it is parallel to either of the rays P/Q or Q/P. Nearly all usual properties of parallel lines are valid, although there is no assumption securing the existence of parallel lines. The last section treats of the extension of the hyperbolic plane by ends defined as pencils of co-parallel lines. Two parallel postulates VI′ and VII are introduced; their independence is shown by examples.

C. Y. Pauc (Cape Town).

Di Noi, Salvatore. Sull'abuso inveterato di un celebre teorema di Euclide. Boll. Un. Mat. Ital. (3) **4**, 79–81 (1949).

Euclid deduced his proposition I.20 (Any two sides of a triangle are together greater than the third side) from I.16 (The external angle at any vertex of a triangle is greater than the internal angle at either of the other vertices). In elliptic geometry I.16 is false (e.g., when one internal angle is obtuse while the other two are right angles). Accordingly, the author offers a new proof of I.20, using instead of I.16 the theorem that the projections of the incenter on the three sides belong to those sides.

H. S. M. Coxeter.

Blaschke, Wilhelm. Zur elliptischen Geometrie. Arch. Math. **1**, 353–361 (1949).

Let each point (x_0, x_1, x_2, x_3) of elliptic space be represented by a quaternion $X=x_0+x_1i+x_2j+x_3k$ of unit norm [cf. Coxeter, Non-Euclidean Geometry, University of Toronto Press, 1942, p. 149; these Rev. **4**, 50]. Then any two pure quaternions (or vectors) determine a line $[r, r']$, the axis of the half-turn $X\rightarrow -rXr'$. [Thus the author's $[r, r']$ is the reviewer's $\{-r, r'\}$; see Amer. Math. Monthly **53**, 136–146 (1946), in particular, p. 144; these Rev. **7**, 387.] The author makes several new applications of this notation, such as the following condition for a closed simply-connected surface to be simple (schlicht): the normals $[r, r']$ should determine a one-to-one correspondence between the vectors r and r', each of which takes in turn every possible direction.

H. S. M. Coxeter (Toronto, Ont.).

Blaschke, Wilhelm. Nicht-Euklidische Mechanik. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1943**, no. 2, 10 pp. (1944).

The author uses the methods of his recent paper [see the preceding review] to obtain equations of motion for a rigid body in elliptic space.

H. S. M. Coxeter (Toronto, Ont.).

Kowalewski, Gerhard. Lobatschefskijs Herz, eine Rollkurve der nichteuklidischen Geometrie. Věstník Královské České Společnosti Nauk. Třída Matemat.-Přírodověd. **1939**, no. 11, 6 pp. (1940).

When a circle rolls on a fixed circle of the same radius, a point on the circumference of the former describes a cardioid. In hyperbolic geometry the shape changes when the radius increases, until as a limiting case we have "Lobačevskiĭ's heart," where a horocycle rolls on a fixed horocycle. Of course, the curve extends to infinity like the horocycle itself. [When the author calls it "heart shaped" he is thinking of a Euclidean model.] Like the Euclidean cardioid, it is a rational quartic curve. Taking the absolute conic to be $x^2+y^2=z^2$, we may express it as the locus of the point $(2s^2, 2s^3-s^4, 2+s^4)$.

H. S. M. Coxeter (Toronto, Ont.).

Nesterovič, N. M. Geometrical constructions with horocycle-compass and ruler in the Lobačevskiĭ plane. Doklady Akad. Nauk SSSR (N.S.) **66**, 1047–1050 (1949). (Russian)

Adler proved the following theorem. Every geometrical construction of the second degree can be performed with a rectangular drawing instrument. The rather unexpected equivalence of the complex right angle-ruler with the classical ruler and compass obtains an explanation when one considers the Lobačevskiĭ plane. In an earlier paper [C. R. (Doklady) Acad. Sci. URSS (N.S.) **43**, 186–188 (1944); these Rev. **7**, 22] the author proved that all geometrical constructions in the L-plane can be performed with ruler and Euclidean compass. In this paper it is proved that (1) the compass can be replaced as a drawing instrument for second degree constructions by a horocycle instrument; (2) since in the limiting case the horocycle degenerates to a right angle Adler's theorem is obtained.

H. A. Lauwerier (Amsterdam).

Hazanov, M. B. On areas in the Lobačevskiĭ plane. Doklady Akad. Nauk SSSR (N.S.) **66**, 571–574 (1949). (Russian)

The author gives a brief survey of the theory of area in the Lobačevskiĭ plane. First angular figures consisting only of two straight lines are considered. Then the author investigates the area of triangles and of figures composed of a finite number of triangles. Finally more general figures with curved boundaries are considered.

H. A. Lauwerier (Amsterdam).

Jordan, P. Über eine nicht-desarguessche ebene projektive Geometrie. Abh. Math. Sem. Univ. Hamburg **16**, 74–76 (1949).

The author has constructed an algebra $\mathfrak{M}_3^8$ [Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. **1932**, 569–575; **1933**, 209–217] whose elements are matrices of order three with entries from the real Cayley number system. Addition and scalar multiplication are ordinary, but a product AB $(=BA)$ is $\frac{1}{2}(A\times B+B\times A)$ where $A\times B$ is the matrix product. In this algebra every idempotent e except 0 or 1 may be given a dimension one or two. An idempotent e of dimension two may be written as a sum of two orthogonal idempotents of

dimension one: $e=e_1+e_2$, $e_1e_2=e_2e_1=0$. The idempotents of dimension one may be taken as points and those of dimension two as lines to form a projective plane. The incidence rule is that the point e_1 lies on the line e if $e_1e=e_1$. The resulting plane is the plane given by R. Moufang [Abh. Math. Sem. Hamburg. Univ. 9, 207–222 (1933)] in which the theorem of the complete quadrilateral holds. No proofs are given for any of these results. The line of thought is clearly an analogue of von Neumann's representation of the subspaces of a continuous geometry by the lattice of principal ideals in a regular ring [Proc. Nat. Acad. Sci. U. S. A. 23, 16–22 (1937)]. *M. Hall* (Columbus, Ohio).

Deaux, R. Sur un théorème de Zeeman. Nieuw Tijdschr. Wiskunde 36, 225–237 (1949).

Making use of the projective properties of conics the author furnishes a synthetic proof of the theorem, due to Zeeman: in a complete quadrilateral with vertices at a finite distance, if one side is parallel to the Euler line of the triangle formed by the remaining three sides, the same holds for all four sides of the quadrilateral. A projective definition of the Euler line of a triangle enables the author to formulate a projective generalization of Zeeman's theorem.

N. A. Court (Norman, Okla.).

Postma, D. The figure of the centers of similitude of four circles. Simon Stevin 26, 149–167 (1949). (Dutch)

In this paper the following basic theorem is proved. (1) The figure of the centers of similitude of four circles is in principle a central projection of a parallelepiped P. (2) The centers of three of the four circles are the vanishing points of the edges and the center of the fourth circle is the projection of the center of P. (3) Each center of similitude is either the projection of the intersection of two face diagonals or the vanishing point of a face diagonal. (4) The projection of each of the face diagonals passes through a center of similitude of two circles and through the center of similitude of the other two circles. The proof is based on stereometrical arguments. Starting from this theorem a detailed survey of the properties of the complete figure is given. *H. A. Lauwerier* (Amsterdam).

Stojaković, Mirko. Sur une propriété d'orthocentre et son application à la résolution de quelques problèmes de la planimétrie. Bull. Soc. Math. Phys. Serbie 1, 51–54 (1949). (Serbian. Russian and French summaries)

Gambier, Bertrand. Courbes algébriques de classe p dont une p-sectrice de chaque système de tangentes concourantes passe par un point fixe. Ann. Sci. École Norm. Sup. (3) 66, 1–17 (1949).

This is a continuation of an earlier study [Bull. Sci. Math. (2) 71, 232–246 (1 plate) (1947); these Rev. 10, 139]. In the present case the fixed point O lies at a finite distance. The equations of the curves considered have real coefficients, if the axes of reference are real. It is shown that if a curve satisfies the imposed conditions (a) the isotropic lines passing through the fixed point O touch the curve at the circular points at infinity and are the only isotropic tangents of the curve; (b) the parallel curves of such a curve are also solutions of the problem; (c) if such a curve is unicursal, its evolute is also a solution; (d) the maximum degree of the curve is $p(p-1)$, the degree of a unicursal curve may be as low as $2(p-1)$, and considerably lower for some. Cubic and quartic curves receive special attention. Calculations are held down to a minimum. *N. A. Court.*

Guillemin, M., et van Wynsberghe, R. Solutions par la règle et le compas de quelques problèmes de géométrie descriptive relatifs à l'hyperboloide à une nappe. Mém. Soc. Roy. Sci. Liége. Coll. in-4°. Sér. 1. 2, no. 1, 22 pp. (1949).

A hyperboloid of one sheet is represented in map projtion (projection cotée). A number of elementary problems such as intersection of the hyperboloid with a plane or with a straight line and construction of tangent planes are solved graphically. *E. Lukacs* (China Lake, Calif.).

Convex Domains, Extremal Problems

Hadwiger, H. Über Masszahlen und Ungleichungen bei Mittelpunktseikörpern. Monatsh. Math. 53, 132–137 (1949).

Let K be a smooth convex body with center O and ρ a positive number. The author defines a body $\bar{K}_\rho$ (not necessarily convex) containing K, and also the outer parallel body K_ρ of K, in the following way. Let P be a boundary point of K and ω_ρ the plane parallel to and at distance ρ from the plane of support at P. The line OP intersects ω_ρ at P_ρ. Then $\bar{K}_\rho$ is the union of all segments OP_ρ. Like the volume of K_ρ, the volume of $\bar{K}_\rho$ is a polynomial of third degree in ρ. Its coefficients V (volume of K), F (surface area of K), $\bar{M}$, $\bar{C}$ are invariants corresponding to the invariants V, F, M, $C=4\pi/3$ occurring as coefficients in the volume of K_ρ. From simple integral representations the inequalities $\bar{M}^2\leqq 3\bar{C}F$ and $F^2\leqq 3\bar{M}V$ (analogous to the Minkowski inequalities but reversed) are easily derived.

W. Fenchel (Los Angeles, Calif.).

Besicovitch, A. S. A variant of a classical isoperimetric problem. Quart. J. Math., Oxford Ser. 20, 84–94 (1949).

This is an extension of Bonnesen's well-known isoperimetric inequality $l^2-4\pi a\geqq(l-2\pi r)^2$ concerning convex domains to arbitrary domains of area a and in-circle radius r bounded by a simple curve of length l. Previous results in this direction were obtained by Turán, and Grünwald and Vázsonyi [Acta Litt. Sci. Szeged 8, 236–240 (1937)] and Santaló [Revista Unión Mat. Argentina 10, 155–162 (1945); these Rev. 7, 168]. Santaló proves (with regard to generalisations to elliptic and hyperbolic geometries) only the weaker inequality $a\leqq rl$. But the reviewer of Santaló's paper pointed out [loc. cit.] that Santaló's proof also yields the above inequality. On the other hand, the author's present proof is more elementary and implies also the determination of the class of all curves for which equality is attained.

L. Fejes Tóth (Budapest).

Orts, J. M.ª. On the estimation of the isoperimetric deficit of some convex curves. Revista Mat. Hisp.-Amer. (4) 8, 291–297 (1948). (Spanish)

It is observed that for a few elementary convex curves $r=r(\theta)$ the estimate $(L^2-4\pi A)/(2A)\leqq\int(r'/r)^2d\theta$ is better than would be obtained from the maximal isoperimetric deficit in an annulus. *L. C. Young* (Madison, Wis.).

Rado, R. Some covering theorems. I. Proc. London Math. Soc. (2) 51, 232–264 (1950).

From a finite or infinite set of congruent homothetic triangles the area of the union of which is A ($<\infty$) there always can be selected a subset containing no two overlapping triangles having a total area at least $A/6$. This and

a number of analogous inequalities (concerning sets of equal circles, homothetic cubes in n dimensions, etc.) are systematically derived by means of general methods which rest on the linear combination of convex bodies and some elementary results of the geometry of numbers.

L. Fejes Tóth (Budapest).

Differential Geometry

Bol, Gerrit. Zur Definition von Schmiegfiguren in der Differentialgeometrie. Arch. Math. 1, 362–370 (1949).

It is well known that the osculating plane (sphere) of a twisted curve as well as the osculating conic section of a plane curve may be obtained by a limiting procedure applied to points which define uniquely the (osculating) figure. On the other hand, the same osculating figure may be obtained by requiring it to have best possible contact with the curve. There are some cases where the second method still leads to a result, while the first method is either not admissible or leads to another result. This is, for instance, the case of the osculating quadric of a tape [see the same vol., 192–199 (1948); these Rev. 10, 402] which in general cannot be obtained by limiting procedures applied to four tangential elements. The author also gives other examples of this type.

V. Hlavatý (Bloomington, Ind.).

Pedrazzini, Pierino. Sulla curva di inseguimento di una curva a doppia curvatura. Period. Mat. (4) 27, 99–103 (1949).

Arghiriade, E. Sur le point principal des courbes tracées sur une surface. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 6, 189–194 (1949).

Consideration is given to two curves C, $\bar{C}$ on a surface S having contact of order n at a point M, not tangent to an asymptotic curve, and with distinct osculating planes at M. The other possible cases were treated by Palozzi [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (6) 7, 321–325 (1928)] and E. Bompiani [Mem. Accad. Sci. Ist. Bologna. Cl. Sci. Fis. (8) 3, 35–38 (1926)]. Bompiani has shown in the case being considered that the principal plane of C, $\bar{C}$ is the tangent plane to S at M. The curves C, $\bar{C}$ are projected from a point O in the tangent plane onto one of the faces of a conveniently chosen tetrahedron, Γ, $\bar{\Gamma}$ being the resulting projections. These latter curves have second order contact at M. Conditions on O in order that Γ, $\bar{\Gamma}$ have third, fourth, and fifth order contact are formed, the principal point being thus located. Some results may be stated as follows. Two curves C, $\bar{C}$ on a surface S, tangent at a point M, with distinct osculating planes π, $\bar{\pi}$ at M, have the same principal point as the sections of S by π, $\bar{\pi}$. Let N, $\bar{N}$ be the transforms of π, $\bar{\pi}$ in the correspondence of Segre. The principal point is the harmonic conjugate of M with respect to N, $\bar{N}$.

V. G. Grove.

Ritter, Robert. Invariante Kennzeichnung der vierten vollständigen Klasse von Biegungsflächen. Arch. Math. 1, 418–426 (1949).

As far as the problem of applicability is concerned, there are four distinct categories of surfaces [cf. Hlavatý, Differentialgeometrie der Kurven und Flächen und Tensorrechnung, Noordhoff, Groningen-Batavia, 1939, pp. 281–313]. On the other hand the construction of a surface from a given metric tensor requires the integration of an invariant partial differential equation, say E, of the second order [Hlavatý, loc. cit., pp. 429–434]. The author states that starting with E one gets another set $S_{II,III}$ of three invariant partial differential equations which characterize the categories II, III and a set S_{IV} of three invariant partial differential equations which characterize the category IV. In all these cases the corresponding canonical form of the metric form is found. A typical surface of the category IV is

$$\mathfrak{y}=p\mathfrak{x}+\int \mathfrak{x}\times(\mathfrak{n}\times d\mathfrak{x}),$$

where $\mathfrak{x}$ is a minimal surface and $\mathfrak{n}$ its normal.

V. Hlavatý (Bloomington, Ind.).

van Bouchout, V. A theorem on hexagon-nets. Simon Stevin 26, 143–148 (1949). (Dutch)

Blaschke and Dubourdieu showed [Abh. Math. Sem. Hamburg. Univ. 6, 198–215 (1928)] that a family of curves is part of a regular hexagon-net (angles of 60°) if and only if it is an isothermal system. In this paper the author proves the following more general theorem. A family of curves on a surface together with two arbitrarily chosen systems of isogonal trajectories forms a hexagon-net if and only if it is an isothermal family.

J. Haantjes (Leiden).

Simonart, Fernand. Sur l'équation différentielle quadratique d'un réseau orthogonal et isotherme. Acad. Roy. Belgique. Bull. Cl. Sci. (5) 35, 346–356 (1949).

A surface S, with the equation $z=f(x, y)$, is said to be harmonic if the function $f(x, y)$ is a harmonic function. Haag [Bull. Sci. Math. (2) 65, 100–103 (1941); these Rev. 7, 77] showed that the projection of the asymptotic net on S onto the plane $z=0$ is an isothermally orthogonal net. The purpose of this paper is to throw more light on this theorem by studying the conditions on the coefficients of a binary quadratic differential form for this form to represent an isothermally orthogonal net. The condition on P, in $Pdx+dy=0$, that an integral $U(x, y)$ arising from an integrating factor μ (that is, $\mu(Pdx+dy)=dU$) be such that the curves $U(x, y)=$constant form an isothermally orthogonal net is (1) $(P_{xx}+P_{yy})/(P_x^2+P_y^2)=2P/(1+P^2)$. The orthogonal trajectories, namely the integrals of $dx-Pdy=0$ ($P\neq 0$), also have the same property. That is, the differential equation $dx^2-(P-P^{-1})dxdy-dy^2=0$ determines an isothermally orthogonal net if P satisfies condition (1). In general the quadratic differential equation $dx^2-2Adxdy-dy^2=0$ determines an isothermally orthogonal net if and only if A is a solution of (1). The projection on the plane $z=0$ of the asymptotic net on the surface S, $z=f(x, y)$, has the differential equation $rdx^2+2sdxdy+tdy^2=0$, $r=f_{xx}$, $s=f_{xy}$, $t=f_{yy}$. This net is orthogonal if and only if $r+t=0$, that is, if and only if S is harmonic. It may be shown that, if $r+t=0$ and $A=s/r$, then A satisfies equation (1) above, that is, Haag's theorem is verified. A converse theorem is proved. That is, suppose $dx^2-2Adxdy-dy^2=0$ defines an isothermally orthogonal net. Does there exist a surface S the projection of whose asymptotic net on $z=0$ is that given net? The answer is in the affirmative, the surfaces being found by quadratures.

V. G. Grove (East Lansing, Mich.).

Varga, O. Vektorfelder, deren kovariante Ableitung längs einer vorgegebenen Kurve verschwindet. Hungarica Acta Math. 1, no. 4, 1–3 (1949).

Let $C(x^1=t, x^2=0)$ be a curve embedded in the net $x^1=t$, $x^2=s$. Denote by $\xi^i(t)$ the solution of $\xi^{i\prime}=-\Gamma^i_{j1}(t, 0)\xi^j$, con-

sider t as an auxiliary parameter and denote by $w^i(t, s)$ the solution of $\partial w^i(t, s)/\partial s = -\Gamma^i_{jk} w^j(t, s)$ with the initial condition $w^i(t, 0) = \xi^i(t)$. Hence, for all cases in which $(\partial w^i(t, s)/\partial t)_{s=0} = \partial w^i(t, 0)/\partial t$ the covariant derivative of the vector field $w^i(x^1, x^2)$ vanishes along the curve C. The author uses this device (in a slightly different form) to prove the same statement for an arbitrary $n > 1$. [Reviewer's remark. Let x^i be the coordinate system for which $\Gamma^i_{jk} = 0$ along C ($\Gamma^i_{j1} = \Gamma^i_{(j1)}$), and let w^i be a vector field whose components with respect to the system x^i are constants. Then $\nabla_j w^i = 0$ along C in any coordinate system.] *V. Hlavatý.*

Haack, Wolfgang. Die kubische Indikatrix eines Strahlensystems und ihre Entartungen. Arch. Math. 1, 454–461 (1949).

The neighborhood of the second order of a line of a congruence of lines may be represented by a rational cubic cone with vertex on the line. This cone is the locus of the middle points of the osculating reguli of the ruled surfaces of the congruence through the given line. The improper curve of the cone, called the cubic indicatrix of the congruence, is determined analytically by the vanishing of a ternary cubic form. Ordinarily the indicatrix is a rational cubic curve. The cases in which this cubic degenerates into lines, or into a line and a conic, are considered. In particular, if the indicatrix degenerates into three lines, the cubic cone degenerates into the focal planes and a plane through the middle point of the generator of the congruence.

V. G. Grove (East Lansing, Mich.).

Behari, Ram, and Mishra, Ratan Shanker. Some formulae in rectilinear congruences. Proc. Nat. Inst. Sci. India 15, 85–92 (1949).

The authors' purpose is to apply tensor methods to the study of rectilinear congruences. The vector $v_\beta = \sum_{i=1}^{3} X^i \lambda^i_{,\beta}$ plays a fundamental role in the discussion. Here X^i is a unit vector, normal to the surface of reference S; λ^i is a unit vector along the congruence; and $\lambda^i_{,\beta}$ denotes the derivative of this vector along the surface S. By use of this vector, the authors obtain simplified proofs of the theorems of Beltrami, Malus-Dupin, etc. *N. Coburn.*

Geĭdel'man, R. M. On congruences of circles which reduce to canal surfaces. Doklady Akad. Nauk SSSR (N.S.) 66, 145–147 (1949). (Russian)

A congruence of circles is studied by means of a conformal frame consisting of two points S_0 and S_4 and three mutually orthogonal unit spheres S_1, S_2, S_3 [the reviewer has taken the liberty of changing the notation to a more convenient form]. The infinitesimal displacement of such a frame is defined by $dS_\alpha = \omega_\alpha{}^\beta S_\beta$ ($\alpha, \beta = 0, 1, \cdots, 4$). The ω's are not arbitrary since $S_i S_j = \delta_{ij}$, $i, j = 1, 2, 3$; $S_i S_0 = S_i S_4 = S_0^2 = S_4^2 = 0$ and $S_0 S_4 = 1$. This gives

$$\omega_\alpha{}^\beta = m_\alpha{}^\beta \omega^1 + n_\alpha{}^\beta \omega^3.$$

A congruence of circles which reduces to two families of canal surfaces is called a congruence K and the most general matrix for a K is determined. A cyclic system of Ribaucour, a congruence with ∞^1 orthogonal surfaces, is called a congruence R. If R is a K the two families of canal surfaces are orthogonal and conversely. A congruence C is a congruence K for which $n_1{}^3 = 0$. The focal surfaces of a congruence C are cyclides of the same type.

M. S. Knebelman (Pullman, Wash.).

Laptev, G. F. Invariant construction of the projective differential geometry of surfaces. Doklady Akad. Nauk SSSR (N.S.) 65, 121–124 (1949). (Russian)

Invariant formulas for the geometry of a surface in projective 3-space are obtained by means of the moving trihedral. Invariance means independence of surface coordinates or of projective normal. If an infinitesimal displacement of the trihedral $M_0 M_1 M_2 M_3$ is defined by $dM^K = \omega_J{}^K M^J$, $\omega_J{}^K$ being linear differential forms, then $[\omega_0{}^k \omega_k{}^3] = 0$, $k = 1, 2$, where $[\omega_K{}^L \omega_L{}^J] = D\omega_K{}^J$. Denoting $\omega_0{}^k$ by ω^k, the Fubini metric of the surface is obtained in the form of $\bar{b}_{ij}\omega^i\omega^j\omega^k/a_{ij}\omega^i\omega^j$. Any line of the canonical pencil of projective normals is then expressible in terms of a parameter σ; for $\sigma = 0$ this gives the direction of Wilczynski; for $\sigma = \frac{1}{3}$, the axis of Čech; for $\sigma = 1$, Fubini's normal, etc. In a similar manner the pencil of Darboux surfaces is expressed in terms of a paremeter λ so that $\lambda = 0$ gives the surface of Lie; $\lambda = -\frac{1}{3}$, the surface of Fubini, etc. The method used can be applied to hypersurfaces in projective n-space as well as for Klein spaces.

M. S. Knebelman (Pullman, Wash.).

Bol, Gerrit. Streifen auf einer Fläche im dreidimensionalen Raum. Arch. Math. 1, 371–376 (1949).

In a previous paper [same vol., 192–199 (1948); these Rev. 10, 402] the author established the main projective results of a theory of "tapes" (Streifen). In this paper he states some theorems (without proofs) of the theory of tapes on a surface, concerning (1) a ruled surface through a given tape, (2) the set of all tapes on a surface through a given point. The basic curve of the tape is considered in connection with flecnodal curves, Darboux curves, pangeodesics and so on. Thus for instance a necessary and sufficient condition that the basic curve be a flecnodal curve of a ruled surface which goes through the tape is $\alpha = \text{constant}$. (Any ruled surface through a given tape is described by the ruling which is defined by the points x and $t + \alpha z$; x, t and z having the meaning mentioned in the review cited above.) The last section is devoted to Moutard quadrics along a curve. *V. Hlavatý* (Bloomington, Ind.).

Stakowski, Walter. Zur Geometrie der Raumkurven im vierdimensionalen projektiven Raum. Arch. Math. 1, 377–382 (1949).

Geometrical interpretation of the "Frenet formulae" obtained by the author in a previous paper [same vol., 200–204 (1948); these Rev. 10, 402; where these formulae are labeled (1)]. The main results may be stated as follows. If $b \neq 0$ then $\int b^{\frac{1}{3}} dt$ is the projective arc of the curve C and $a(s)$, $\mu(s)$, $c(s)$ its "projective curvatures." If, and only if, a, b, μ, c are constant for a convenient choice of parameter, C admits a one-parameter group of projective transformations. The author also discusses the geometrical interpretation of vanishing projective curvatures. *V. Hlavatý.*

Villa, M., e Sangermano, C. Condizione affinchè una trasformazione puntuale fra due S_3, in una coppia a jacobiano nullo, sia osculabile con una trasformazione cremoniana. Boll. Un. Mat. Ital. (3) 4, 23–30 (1949).

Given a point-transformation T between two projective three-dimensional spaces, S, $\bar{S}$, it is generally impossible to find a Cremona transformation osculating T in the neighborhood of a pair of corresponding points O, $\bar{O}$ if O is a regular point of the Jacobian of S. This is however possible if and only if the planes through $\bar{O}$ are transformed into surfaces having at O as asymptotic tangent the stationary

tangent. In this case the approximation can be performed by cubic Cremona transformations. *E. Bompiani.*

Rollero, Aldo. Riferimenti intrinseci per lo studio locale delle trasformazioni puntuali fra due S_3. Boll. Un. Mat. Ital. (3) **4**, 45–48 (1949).

The author determines intrinsic projective references at corresponding regular points O, $\bar{O}$ in a point-transformation between two projective 3-dimensional spaces. To this end four inflexional directions (no three coplanar) are assumed as axes x, y, z and as unit-line in the star centered at O (and the corresponding lines at $\bar{O}$). To determine the planes at infinity and the unit-points the caps of the third order in one space corresponding to the coordinate planes in the other space and the characteristic projectivities are used. Another reference was given by M. Villa [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **4**, 55–61, 192–196, 295–303 (1948); these Rev. **10**, 210]. *E. Bompiani.*

Lyukšin, V. S. The embedding of a two-dimensional Riemannian manifold in a three-dimensional Euclidean space. Izvestiya Akad. Nauk SSSR. Ser. Mat. **13**, 363–384 (1949). (Russian)

The author investigates anew a problem solved to one extent or another by Darboux, Janet, and others, of determining all surfaces R_2 in three-space whose metric is given by $ds^2=Edu^2+2Fdudv+Gdv^2$, where E, F, and G are three preassigned functions analytic in some domain of the variables u and v. His principal tool is a basic theorem of J. M. Thomas relating to certain systems of differential equations [Differential Systems, Amer. Math. Soc. Colloquium Publ., v. 21, New York, 1937]. The problem leads to a well-known system of differential equations not of any standard type for existence theorems. The author carries out an explicit step by step modification of the system so that it becomes in turn a simple system in the sense of J. M. Thomas, a standard system, a passive system and, finally, a definite system. At this point the theorem of Thomas guarantees a unique solution for appropriate boundary conditions.

The solutions are analyzed in some detail; they fall into two classes, the general (Σ_1) and the particular (Σ_2). The distinction comes to turn about the question whether the parametric lines $u=c$ and $v=c$ are asymptotic lines (Σ_2) or are not (Σ_1). The general solutions have a pair of arbitrary functions associated with them; these measure the "pliability" of the surface as well as its freedom of position. There are four natural classes of particular solutions, two of them containing surfaces of negative curvature. Here in Σ_{211} the parametric lines $v=c$ are asymptotic but not geodesic, and in Σ_{212} they are asymptotic and also geodesic. The solutions in the class Σ_{221} are doubly asymptotic, in Σ_{222} they are geodesic as well. This last class has an arbitrary function $\varphi(v)$ associated with each solution, measuring the pliability of the surface with preservation of its linear generators. *L. Zippin* (Flushing, N. Y.).

Van Bergen, F. Mouvement d'un solide dans un espace riemannien. Acad. Roy. Belgique. Bull. Cl. Sci. (5) **35**, 234–236 (1949).

The author points out that the correspondence between projective Cayley spaces and Riemannian spaces with constant curvature may be used to apply the method of De Donder to the study of the movement of a solid in a space of constant curvature. [Cf. T. De Donder, same Bull. Cl. Sci. (5) **28**, 8–16, 60–66 (1942); **32** (1946), 295–299 (1947); these Rev. **7**, 176; **9**, 67.] The results are first written in homogeneous form of degree zero, and then the proper normalization is applied. *A. Schwartz.*

Kanitani, Jôyô. Sur l'espace à connexion projective majorante. I. Jap. J. Math. **19**, 343–361 (1947).

L'auteur définit d'abord par la méthode du repère mobile le développement d'une courbe d'une variété R à n dimensions plongée dans un espace projectif S_N à $N>n$ dimensions. Le second chapitre est consacré à la définition du cône des tangentes de Darboux en chaque point. Ce cône (cubique) est d'abord défini d'une manière invariante analytiquement et ensuite par une propriété géométrique. Dans le troisième chapitre sont démontrées certaines propriétés sur le contact des courbes de S_n. Dans S_N il existe toujours une variété à n dimensions S_n ayant un contact du deuxième ordre au point A avec le développement d'une courbe quelconque de S_n. Pour qu'il en existe une qui a un contact du troisième ordre, il faut et il suffit que toute courbe fermée infiniment petite de S_n partant du point A et y retournant soit développée sur une courbe dont l'extrémité soit située dans l'espace tangent à S_n. Deux autres théorèmes sont établies qui donnent des conditions pour un contact du quatrième ordre et du cinquième ordre s'exprimant aussi par des propriétés du développement d'un contour infinitésimal. *M. Haimovici* (Jassy).

Yen, Chih Ta. Sur les systèmes de variétés à k dimensions et la connexion projective normale associée. C. R. Acad. Sci. Paris **228**, 1844–1846 (1949).

This note continues the study in a previous note [same C. R. **227**, 461–462 (1948); these Rev. **10**, 211] of the geometry of systems of k-dimensional varieties in an n-space. An identity for the components of the curvature tensor of an associated normal projective connection is given, from which some consequences are drawn, in particular, necessary and sufficient conditions that the system is equivalent under point transformations to the system of linear spaces in an ordinary projective space. It may be remarked here that the geometry in question has also been studied by the reviewer [Proc. Nat. Acad. Sci. U. S. A. **29**, 38–43 (1943); these Rev. **4**, 259]. *S. Chern* (Chicago, Ill.).

Hlavatý, V. Affine embedding theory. I. Affine normal spaces. Nederl. Akad. Wetensch., Proc. **52**, 505–517 = Indagationes Math. **11**, 165–177 (1949).

L'auteur se propose de construire dans une suite de travaux (dont celui-ci est le premier) une théorie des sous-espaces d'un espace à connexion affine. Dans ce travail, il définit d'abord une connexion affine d'un X_m plongé dans un A_n. Cette définition est intrinsèque si l'X_m est maximal (c'est-à-dire si tous les espaces osculateurs jusqu'à un ordre N sont de dimensions maxima, ceux d'ordre supérieur, étant compris dans celui d'ordre N). Dans ce cas, les composantes de cette connexion requièrent les dérivées de l'ordre $(N+1)$ème des coordonnées de A_n et leurs transformations celles du deuxième ordre des coordonnées de l'X_m. Ensuite il définit des connexions de degré supérieur, pour les dérivées covariantes d'ordre supérieur. Ces connexions sont du même ordre $N+1$ par rapport aux coordonnées de A_n si l'X_m est maximal. Enfin, il définit d'une manière intrinsèque des espaces normaux à la variété X_m. *M. Haimovici.*

Vagner, V. V. On the embedding of a field of local surfaces in X_n in a constant field of surfaces in affine space. Doklady Akad. Nauk SSSR (N.S.) 66, 785–788 (1949). (Russian)

A field of M-dimensional surfaces in an affine E_N is said to be a constant field if the M-space associated with a point P_1 is mapped into that associated with P_2 when P_1 is mapped into P_2 by a translation. The parametric equations of the M-spaces may be given by $X^A=L^A(H^C)+X_0^A$ $(A=1, \cdots, N;$ $C=1, \cdots, M)$ [apparently the author means translation in the metric and not in the affine sense: this presupposes a Euclidean metric in E_N]. If an n-space is given by (a) $X^A=\phi^A(\xi^\alpha)$, $\alpha=1, \cdots, n$, it is assumed that its tangent n-planes intersect the M-surfaces associated with the points of tangency. This gives a field of m-surfaces (of intersection) given by $X^A=\phi_a^A(\xi^\lambda)\cdot c^a(\xi^\lambda, \eta^c)+\phi^A(\xi^\lambda)$, $c=1, \cdots, m$. This is the induced field; it is normally induced if a small deformation of the n-space produces again an induced field in the M-surfaces. In this case $m=M+n-N$. Thus the m-spaces $x^a=c^a(\xi^\lambda, \eta^c)$ constitute a local field in the space defined by (a). The paper is concerned with the converse problem and the existence of a solution of the second order partial differential equations arising therefrom. The main result is the theorem that every field of local m-surfaces in X_n may be embedded in a constant field of $(m+k)$-surfaces of E_{n+k}, where $k \leqq n$. From the above point of view the problem of embedding a Riemann space in a Euclidean one is equivalent to the problem of embedding a field of locally regular central hyperquadrics in X_n, whose centers coincide with the centers of the tangent spaces, in a constant field of hyperquadrics in E_N. *M. S. Knebelman* (Pullman, Wash.).

Ôtsuki, Tominosuke. On a space with affine connection which has no closed path. Tôhoku Math. J. (2) 1, 88–90 (1949).

The author gives an example of an affine connection on the torus which has no closed path. *S. Chern.*

Rozenfel'd, B. A. Unitary-differential geometry of families of K_m in K_n. Mat. Sbornik N.S. 24(66), 53–74 (1949). (Russian)

By a unitary-elliptic n-space K_n is understood a complex space of n dimensions in which there is a real metric, the distance between the points x^i, y^i $(i=0, 1, \cdots, n)$ being given by $\cos^2 \omega/r=x^\lambda \bar{y}^\lambda \cdot y^\alpha \bar{x}^\alpha / x^\lambda \bar{x}^\lambda \cdot y^\alpha \bar{y}^\alpha$, x^i being homogeneous coordinates and r a constant, real or pure imaginary (in this paper $r=1$). The space K_n may also be considered as a real Riemann space of $2n$ dimensions; in the first case its curvature as determined by any 2-dimensional orientation is $1/r^2$, but in the second its curvature is variable. A unitary-elliptic K_1 is isometric with a 2-sphere in E_3 of radius $r/2$; thus real 2-dimensional orientations have curvatures $1/r^2$ and $4/r^2$. A sequence of points in K_n is called a thread and if every quadruple of points on a thread has a real anharmonic ratio, the thread is called a 1-chain or just a chain. Chains lie on unitary-elliptic lines K_1; geodesic threads are special cases of chains. They are represented by those circles which cut orthogonally the absolute of K_n given by $x^\lambda \bar{x}^\lambda=0$ and are called normal chains.

The group of motions of K_n consists of two components; the unitary component of all proper unitary-elliptic motions and one of improper motions. The proper motions of K_n are given by unitary unimodular $(n+1)$-rowed matrices so that if a_j^i is such a matrix so is $e^{2k\pi i/(n+1)} \cdot a_j^i$ $(k=0, 1, \cdots, n)$. Thus the group of proper motions of K_n is simply isomorphic to the factor group of unitary unimodular matrices by its discrete normal divisor consisting of the matrices $e^{2k\pi i/(n+1)}\delta_j^i$. Improper motions consist of products of proper motions and mappings of a point by its complex conjugate. In the correspondence of K_1's to 2-spheres in E_3, proper motions correspond to proper rotations of the sphere, while improper ones contain a reflection through the center.

The author defines an m-plane and its polar $(n-m-1)$-plane and the geometry of these planes is studied by means of the group of reflections in an m-plane. An n-chain is defined as an n-parameter K_n in which any two points determine a 1-chain; they are analogues of totally geodesic varieties. Reflections in n-chains are also studied. Another topic studied in detail is that of congruences and pseudo-congruences of m-planes and n-chains. By using the fact that the group of motions of K_n is a compact simple Lie group, one introduces the Cartan metric in the group manifold and by means of it the author obtains the various components of affine curvature. The paper is concluded by a construction of a real model of a unitary-elliptic space K_n in which an m-plane is represented by a real S_m in an S_n, together with an extensive list of their corresponding properties. *M. S. Knebelman* (Pullman, Wash.).

Norden, A. Conformal interpretation of a Weyl space. Mat. Sbornik N.S. 24(66), 75–85 (1949). (Russian)

By a normalized surface X_m in a conformal Möbius space of n dimensions is understood a surface at each point of which there is a normalizing sphere S_{n-m} orthogonal to X_m. On each normalized X_m is determined an affine connection which defines the intrinsic geometry of X_m. To each direction in X_m one can assign a hypersphere of support which is orthogonal to this direction and contains the normalizing sphere. In case $m=n$, with a point M is associated a point N, both being on the supporting sphere. Using polyspherical coordinates, X_m is given by $x=x(u^1, u^2, \cdots, u^m)$ and it is sufficient to prescribe the associated point X in order to determine the normalizing sphere S_{n-m}. This sphere may also be determined by m supporting hyperspheres given by $y_i=\partial_i x-l_i x$, with a proper choice of l_i. Since with each point may be associated $n+2$ independent hyperspheres x, X, y_i, ξ_r $(r=1, \cdots, n-m)$, ξ_r being hyperspheres orthogonal to m independent directions on S_{n-m}, one may express $\partial_j y_i$ in terms of these hyperspheres. This leads to $dx^2=g_{ij}du^i du^j$ as the angular metric of the surface X_m from which one obtains $\nabla_j y_i=l_j y_i-g_{ij}X-p_{ij}x$ and $\nabla_k g_{ij}=2l_k g_{ij}$, thus showing that X_m is a Weyl space with an angular metric induced by the metric of the Möbius space. If $x=x(u^1, \cdots, u^n)$ and $u^i=u^i(p^\alpha)$ $(i=1, \cdots, n;$ $\alpha=1, \cdots, m<n)$, $\partial u^i/\partial p^\alpha=A_\alpha^i$, then the metric induced in the subspace is given by $g_{\alpha\beta}=A_\alpha^i A_\beta^j g_{ij}$, $p_{\alpha\beta}=A_\alpha^i A_\beta^j p_{ij}$, $l_\alpha=A_\alpha^i l_i$ so that the intrinsic geometry of the subspace is induced by that of the enveloping space.

The main part of the paper is devoted to special normalizations. An involutional normalization is one for which p_{ij} is symmetric, which implies that l_i is a gradient; therefore a canonical normalization may be chosen in which $l_i=0$ so that $\nabla_k g_{ij}=0$, $p_{k(ij)}=0$, $\frac{1}{2}R_{kji}^l=-g^{rl}p_{r[k}g_{j]i}-p_{ij}\delta_{[k}^l$. If X is inverse to x with respect to a fixed hypersphere A, then $X=fx+A$ and normalizing x so that $Ax=1$ we have $f=-A^2/2$. In this case we have $\nabla_k g_{ij}=0$ and $R_{kji}^l=-4f\delta_{[k}^l g_{j]i}$, which shows that the space is one of constant curvature. If the inversion is with respect to the absolute the spaces are called equiabsolute and their metric is always Riemannian. It is shown that for such a space there exists a

parametrization in which

$$ds^2 = g_{\alpha\beta}du^\alpha du^\beta + F(u^1, \cdots, u^{n-k})\gamma_{\sigma\rho}du^\sigma du^\rho,$$

where $g_{\alpha\beta}$ depend on $u^1, \cdots, u^{n-k}$ and $\gamma_{\sigma\rho}$ on $u^{n-k+1}, \cdots, u^n$. *M. S. Knebelman* (Pullman, Wash.).

Kosambi, D. D. Lie rings in path space. Proc. Nat. Acad. Sci. U. S. A. **35**, 389–394 (1949).

Verf. geht aus von dem durch

(1) $$\ddot{x}^i + \alpha^i(x, \dot{x}, t) = 0, \qquad i = 1, 2, \cdots, n; \ \dot{x}^i = dx^i/dt,$$

bestimmten Raum der Bahnen. Bei Benützung der Abkürzungen $\varphi_{,r} = \partial\varphi/\partial x^r$; $\varphi_{;r} = \partial\varphi/\partial\dot{x}^r$; $\dot{\varphi} = \partial\varphi/\partial t + \varphi_{,r}\dot{x}^r - \varphi_{;r}\alpha^r$ sind die zu der infinitesimalen Abänderung $\bar{x}^i = x^i + u^i\delta t$ gehörigen Variationsgleichungen von (1) durch

(2) $$\Theta u^i = \ddot{u}^i + \alpha^i_{;r}\dot{u}^r + \alpha^i_{,r}u^r = 0$$

bestimmt. Verf. ordnet jeder Lösung u^i von (2) den Operator

(3) $$X = u^r\partial/\partial x^r + \dot{u}^r\partial/\partial\dot{x}^r$$

einer einparametrigen Lieschen Gruppe des $(2n+1)$-dimensionalen Raumes V_{2n+1} mit Koordinaten $(x, \dot{x}, t)$ zu. Es wird nun untersucht, wann die zu verschiedenen Lösungen von (1) gehörigen Operatoren (3) eine Liesche Gruppe bilden und über welchen Bereich. Dazu genügt es zu zeigen, dass der Klammerausdruck von zwei Operatoren des Typus (3) zu einem Operator desselben Typus führt, und ferner, dass für drei solche Operatoren die Jacobische Identität gilt. Mittels des Hilfssatzes, dass die Lösungen von (2) gerade diejenigen Funktionen sind, deren zugeordnete Operatoren (3) mit der Operation d/dt vertauschbar sind, erhält Verf. folgendes Resultat: Die Operatoren, die zu den Lösungen (2) gehören, bilden einen Lieschen Ring über der Menge derjenigen Funktionen, die längs den Bahnen (1) konstant sind, und bilden eine Liesche Algebra über dem Körper der reellen Zahlen. Verf. untersucht ferner diejenige Untergruppe, die den Urraum x^i und die Bahnen invariant lässt, und findet, dass eine solche Gruppe höchstens $n(n+1)$-gliedrig sein kann. Die Untersuchungen werden weiter auf Systeme von Differentialgleichungen höherer Ordnung und Systeme von partiellen Differentialgleichungen zweiter Ordnung ausgedehnt. Schliesslich beweist Verf. für den Fall, dass eine Metrik für den Raum der Bahnen existiert (d.h. dass (1) die Extremalen eines regulären Variationsproblems darstellt), dass eine Metrik in eine Metrik durch diejenige Gruppe überführt wird, die den Urraum und die Bahnen invariant lässt. *O. Varga.*

Busemann, Herbert. Angular measure and integral curvature. Canadian J. Math. **1**, 279–296 (1949).

Contrary to the various attempts in defining the angular metric in a Finsler space, the author adopts a geometrical approach and defines the angular measure in terms of a system of S-curves, curves which share some of the properties of geodesics. Integral curvature is then defined in terms of the excess of the sum of angles of a geodesic triangle over π. These concepts turn out to be fruitful. After a discussion of some results on excess and parallels the author derives for Finsler geometry the classical Gauss-Bonnet theorem and a major part of Cohn-Vossen's results on the integral curvature of a noncompact surface. While some of Cohn-Vossen's arguments are reproduced, the treatment is considerably simplified. *S. Chern* (Chicago, Ill.).

NUMERICAL AND GRAPHICAL METHODS

***Comrie, L. J. Chambers's Six-Figure Mathematical Tables. Vol. I. Logarithmic Values.** D. Van Nostrand Company, Inc., New York, N. Y., 1949. xxii+576 pp. $10.00.

***Comrie, L. J. Chambers's Six-Figure Mathematical Tables. Vol. II. Natural Values.** D. Van Nostrand Company, Inc., New York, N. Y., 1949. xxxvi+576 pp. $10.00.

These two volumes constitute a compendium of more than 50 tables of elementary and nearly elementary mathematical functions given mostly to 6 decimals. These are no mere handbook tables. The argument intervals are fine enough to render the tables linear in almost all cases. First differences are given almost universally. Near the singularities of functions, auxiliary tables are given which permit easy methods of evaluation. The typography is excellent throughout. The reader cannot fail to notice, and benefit by, the almost entire absence of rulings. The first volume is intended largely for those computers to whom a desk calculator is not available. The second volume will prove more useful to those more fortunate computers with access to a machine. Either volume, however, can be used with only pencil and paper, and both are available to the machine user. The reader is warned that in v. I a table headed $f(x)$ really gives $\log_{10} f(x)$ in most cases. Table IX entitled the "Gamma or factorial function" really gives $\log_{10}\Gamma(x)$. In fact the reader will do well to consult the introduction before using any of the tables. Here he will often find invaluable advice in the use of the table, based on extensive experience.

A short bibliography of similar tables is given at the end of each volume. The main tables of the two volumes are briefly as follows [for a complete and precise description of all tables see Math. Tables and Other Aids to Computation **3**, 465–466 (1949)]. v. I. Common logarithms and antilogarithms, logarithms of trigonometric functions at intervals of one minute and one hundredth of a degree, logarithms of the hyperbolic functions at intervals of .001. Tables of "proportional parts," giving the first nine multiples of the integers less than 1000. v. II. The six trigonometric functions in the first quadrant at intervals of one minute, one hundredth of a degree, and one thousandth of a radian. Exponential and hyperbolic functions (100 pages), natural logarithms, inverse circular and hyperbolic functions, the Gudermannian and its inverse. Powers, roots, reciprocals and factorials of the first 1000 integers. Factor tables to 3400 and list of primes to 12919. Gamma function, probability functions. Volume II closes with a wealth of formulas and tables for the practical application of numerical methods.

The author's meticulous care in the preparation of the two volumes is evident. Numerous errata lists in other tables have appeared during the last few years as the result of collating them with the proof sheets of the present work. There is no doubt that these tables live up to the usually high standards of accuracy and typographical perfection already well established by the author. *D. H. Lehmer.*

Tedeschi, Bruno. Sulla ricerca del tasso col metodo del Sonnet. Boll. Un. Mat. Ital. (3) **4**, 57–61 (1949).

The equation $m = (1+i)^n(1+pi)$ is put in the form $(1+i)^n = m/(1+pi)$ and then the rate of interest i can be found by iteration. *P. Johansen* (Copenhagen).

Porter, A., and Mack, C. New methods for the numerical solution of algebraic equations. Philos. Mag. (7) **40**, 578–585 (1949).

For the case when a large number of real quartics having similar coefficients have to be solved the decomposition of the quartics into two quadratic factors is superior to other methods. The authors recommend it and use chiefly two methods according as all roots have real parts of the same or of different sign. The methods are extended to equations of the fifth and sixth degree. In a third method the remainder $gx+h$ of $f(x)$ after division by x^2+ax+b is studied for several a_i with b=constant and for several b_i with a=constant. The intersection of two smoothing curves gives an approximate value for the true a and b. *E. Bodewig.*

Moore, Edward F. A new general method for finding roots of polynomial equations. Math. Tables and Other Aids to Computation **3**, 486–488 (1949).

Since the existing methods of solving algebraic equations require an intellect capable of choosing and changing the method of calculation, the author draws the attention to the necessity of having a method by which a machine can automatically solve such equations. He also indicates such a method, adding that it is still unsatisfactory.

E. Bodewig (The Hague).

Isakson, Gabriel. A method for accelerating the convergence of an iteration procedure. J. Aeronaut. Sci. **16**, 443 (1949).

When x_i form a sequence obtained in an iteration procedure the author rediscovers the well-known formula $X=\lim x_i=x_3-(x_3-x_2)^2/(x_3-2x_2+x_1)$. [The author seems to overlook the fact that this formula, which was first established by Aitken [Proc. Roy. Soc. Edinburgh **57**, 269–304 (1937)], is valid only if the convergence of the sequence is linear. On the other hand, and in contradiction to the author's statement, the formula is also valid if the x_i oscillate about X, as may be seen from the definition of linear convergence or by replacing the author's starting point $x_n=X+ce^{-kn}$ by $x_n=X+cq^n$, where $q\neq 0$.]

E. Bodewig (The Hague).

Herzberger, M. The normal equations of the method of least squares and their solution. Quart. Appl. Math. **7**, 217–223 (1949)

The paper treats the normal equations by vector notation. The known methods of solution (decomposition of the matrix into two triangular matrices) are derived by using orthogonal vectors. [Concerning details the reviewer remarks that Schur's paper cited has nothing to do with the square-root method, which was first described by Choleski; reference no. 4 is incorrect; there are slight misprints in equations (24) and (33) and adjacent lines.]

E. Bodewig (The Hague).

Verzuh, Frank M. The solution of simultaneous linear equations with the aid of the 602 calculating punch. Math. Tables and Other Aids to Computation **3**, 453–462 (1949).

This is a rediscovery of Aitken's well-known condensation method [Proc. Roy. Soc. Edinburgh. **57**, 172–181 (1937)] for the special case of linear equations. The evaluation is performed by means of the 602 calculating punch. The author emphasizes that the error of the successive eliminations is small and that two extra figures will frequently suffice to eliminate its influence in practice. *E. Bodewig.*

Greenwood, Robert E., and Danford, Masil B. Numerical integration with a weight function x. J. Math. Physics **28**, 99–106 (1949).

Numerical methods of integration of a function multiplied by a weighting factor x are presented for the intervals 0 to 1 and -1 to 1. The formulas are similar to Chebyshev's in that the integration is performed by multiplying the sum of a series of values of the function by a single constant. Such a method is particularly useful where the probable errors of terms in the series are all equal. Both Chebyshev's method and a new method are used to determine the values of constants used in the integration. In the case of integration over the interval 0 to 1, 1 point, 2 point, and 3 point formulas are presented, and it is shown that formulas having 4 to 10 points are not possible. For integration over the interval -1 to 1 formulas with 2, 4, 6, and 8 points are given. In a numerical example for which the function is e^x, it is found that the new integration formulas are substantially better than either the Newton-Cotes or Chebyshev formulas. *S. Levy* (Washington, D. C.).

Loud, Warren S. On the long-run error in the numerical solution of certain differential equations. J. Math. Physics **28**, 45–49 (1949).

The author considers the accuracy of numerical methods of solving equation (1) $y'=f(x, y)$ but the discussion is confined to (2) $y'=ay$ and the generalization to (1) is not clear. The numerical technique of solving (2) is assumed to be an rth order linear difference equation with constant coefficients linking ordinates y_i at equidistant x-interval h. For a large number of (say) n steps the solution will be approximately of the form cf_r^n, where f_r is an rth order Taylor approximation to e^{ah}. From this an approximate relative error, a constant multiple of $n(ah)^{r+1}$, is derived. Since no attempt is made to consider other sources of error (e.g., rounding-off errors) the conclusions based on the above expression of "truncation error" (or better, remainder term error) are practically limited as the latter would normally be a small fraction of the rounding-off error at each stage of the procedure. *H. O. Hartley* (London).

Salvadori, Mario. Sul calcolo degli autovalori mediante differenze finite ed estrapolazione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **6**, 181–183 (1949).

When the differential operator in a linear differential eigenvalue problem is replaced by differences with the interval h converging to 0, Richardson [Philos. Trans. Roy. Soc. London. Ser. A. **210**, 307–357 (1911)] made the assumption that the error e_λ of the eigenvalue can be developed in the form $e_\lambda=ah^2+bh^4+\cdots$ ($a, b, \cdots$=constants) and used this for improving the values of λ yielded by several systems of linear equations. The author now proves that this assumption is useful, but not sufficient. Then he states empirically that the extrapolations of Richardson will give the best results when applied to a monotonic series of successive approximations to λ. *E. Bodewig* (The Hague).

Bückner, H. Über Konvergenzsätze, die sich bei der Anwendung eines Differenzenverfahrens auf ein Sturm-Liouvillesches Eigenwertproblem ergeben. Math. Z. **51**, 423–465 (1948).

On considère le problème (identique à son adjoint) $[r(x)y'(x)]'+[\lambda p(x)+q(x)]y(x)=0$, $y(a)=y(b)=0$, avec $p(x)>0$, $r(x)>0$. On donne un procédé aux différences pour le calcul approché des valeurs propres $\lambda_1<\lambda_2<\cdots$ et des correspondantes solutions propres $y_1(x), y_2(x), \cdots$. Si n est un entier $\geqq 2$ et $f(x)$ une fonction arbitraire définie dans (a, b), on employera les notations suivantes: $h=(b-a)/n$; $x_i=a+ih$ $(0\leqq i\leqq n)$; $f(x_i)=f_i$; $\Delta f_i=f_{i+1}-f_i$ $(0\leqq i\leqq n-1)$. Dans la n-iéme approximation on écrira $\Lambda^{(n)}$ au lieu de λ. Pour construire cette approximation on doit procéder comme il suit. On fixe arbitrairement quatre suites uniformement convergentes de fonctions continues dans (a, b): [1] $\sigma^{(n)}(x)$, $\tau^{(n)}(x)$, $\xi^{(n)}(x)$, $\zeta^{(n)}(x)$. En posant

$$r(x)=r(x)+h^2[\Lambda^{(n)}\sigma^{(n)}(x)+\tau^{(n)}(x)], \quad \bar{p}(x)=p(x)+h^2\xi^{(n)}(x),$$
$$\bar{q}(x)=q(x)+h^2\zeta^{(n)}(x),$$

on écrit ensuite, pour les valeurs approchées F_i de $y(x_i)$, le système suivant d'équations linéaires homogènes: [2] $\Delta(\bar{r}_{i-1}\Delta F_{i-1})+h^2(\Lambda^{(n)}\bar{p}_i+\bar{q}_i)F_i=0$ $(i=1, 2, \cdots, n-1)$; [2'] $F_0=F_n=0$. Dans les $n-1$ équations [2], avec le inconnues $F_1, \cdots, F_{n-1}$, le paramètre $\Lambda^{(n)}$ parait linéairement. On déduit notamment pour $\Lambda^{(n)}$ une équation algébrique de degré $n-1$, dont les racines $\Lambda_1^{(n)}\leqq\Lambda_2^{(n)}\leqq\cdots\leqq\Lambda_{n-1}^{n}$, qui sont toutes réelles, donnent lieu à des solutions propries $F_i^{(n,k)}$ $(k=1, 2, \cdots, n-1)$ du système [2].

Il faut observer que si les fonctions [1] sont nulles, la méthode se réduit à celle de Collatz [Deutsche Math. **2**, 189–215 (1937)]; l'introduction de ces fonctions donne une amélioration des théorèmes de convergence pour $n\to\infty$. On démontre, par exemple, avec des convenables hypothèses sur les fonctions [1], que la limite $\lim_{n\to\infty}(\lambda_k-\Lambda_k^{(n)})/h^2$ est finie; on peut aussi choisir les [1] de sorte que $\lim_{n\to\infty}(\lambda_k-\Lambda_k^{(n)})/h^4$ soit finie. De plus, les fonctions polygonales déterminées par les points $[x_i, F_i^{(n,k)}]$ convergent uniformement vers la solution propre $y_k(x)$, etc. Des cas particuliers et des exemples numériques sont étudiés. Les résultats sont obtenus après une étude des propriétés (extrémales, de orthogonalité, etc.) des $\Lambda_k^{(n)}$ et $F_i^{(n,k)}$.

A. Ghizzetti (Pisa).

Frocht, M. M. Some simplifications in the numerical solution of Laplace's equation with special applications to photoelasticity. Proc. Soc. Exper. Stress Analysis **6**, 39–43 (1948).

The author is concerned with the numerical solution of Laplace's equation $\Delta u=0$ for prescribed boundary values u_Γ (on the boundary Γ of a plane region R) by the well-known iteration method: R is covered by a rectangular grid (i, j) and the values of the solution u_{ij} at the grid points linked by the finite difference analogue to

$$(1) \qquad 4u_{i,j}=u_{i+1,j}+u_{i-1,j}+u_{i,j+1}+u_{i,j-1},$$

which is solved by iteration subject to the prescribed interpolates u_Γ on Γ. The author describes a graphical method of obtaining initial values of u_{ij}. These are obtained by passing a "rosette" of lines through i, j and taking as an initial value the weighted averages of the boundary values at the first intersections of the rosette lines with Γ, the weights being the reciprocals of the distances of i, j from these points of intersection. The method is illustrated with examples. *H. O. Hartley* (London).

Bruk, I. S. An electric minimizer. Doklady Akad. Nauk SSSR (N.S.) **62**, 481–484 (1948). (Russian)

The apparatus, of which no design details are given, minimizes one or the other of the expressions

$$H(X)=\sum_{i=1}^{n}\left(\sum_{k=1}^{n}a_{ik}x_k-b_i\right)^2,$$

$$\varphi(X)=\sum_{i=1}^{n}\left|\sum_{k=1}^{n}a_{ik}x_k-b_i\right|$$

by means of successive alterations of the vector $X=\{x_j\}$. The overall circuit differs but little from that used by F. J. Murray [pp. III–18 of the book reviewed below] to minimize $H(X)$. In the present case the sum of squares is obtained by impressing the voltage for each linear expression on a corresponding resistance arranged so that the heat produced activates one of a series of n thermocouples. An alternative arrangement using rectifiers is provided for $\varphi(X)$. The paper states [for proof in the case of $H(X)$ see Murray, loc. cit.] that no conditions are needed on $A=\{a_{ik}\}$ for convergence of the process when the solution is unique. Automatic operation in the case of $\varphi(X)$ is briefly discussed. It is said to be possible to adapt the apparatus to the solution of $|A-\lambda I|=0$ without serious change. Also, by replacing the integrals involved by quadrature formulas, it can be applied to the problems formulated by L. V. Kantorovitch [C. R. (Doklady) Acad. Sci. URSS (N.S.) **48**, 455–460 (1945); these Rev. **8**, 30] in terms of minimizing.

R. Church (Annapolis, Md.).

***Murray, Francis J. The Theory of Mathematical Machines.** 2d ed. King's Crown Press, New York, N. Y., 1948. ix+139 pp. $3.00.

For a detailed review of the first edition [1947] see these Rev. **9**, 103. The revision of this work has added 23 pages, mostly on electronic digital computers. There is also a short new section on noise and stability. The usefulness of the book as a reference work has been increased by the addition of an author-subject index. *D. H. Lehmer.*

Booth, Andrew D., and Britten, Kathleen H. V. Principles and progress in the construction of high-speed digital computers. Quart. J. Mech. Appl. Math. **2**, 182–197 (1949).

The authors give a brief summary of some, at least, of the basic principles which underlie the design of high-speed digital computers. Among other topics they include a discussion of number bases, serial versus parallel operation, the logical functions of the control organ, the code and various types of storage devices. In the second half of the paper the authors discuss some of the present-day developments in the U. S. A., including the Harvard Marks I and II, the Bell relay computers, the University of Pennsylvania EDVAC, the Institute for Advanced Study machine and the MIT machine. Here they describe briefly the status of each project as of 1947. [In the case of the Institute machine, the reviewer wishes to correct some points in the paper. It is not intended that the Institute machine multiply in less than 100 μs nor add in less than 2 μs. It would be more realistic to increase these numbers by a factor between 3 and 5.] *H. H. Goldstine* (Princeton, N. J.).

RELATIVITY

Einstein, A., and Infeld, L. On the motion of particles in general relativity theory. Canadian J. Math. 1, 209–241 (1 plate) (1949).

The authors' philosophical standpoint is that the gravitational field is the fundamental physical reality, manifesting itself in the motion of bodies, not that the field is merely a useful mathematical device for describing the motion of bodies, regarded as fundamental. Hence it is a problem of significance to derive the equations of motion from the field equations and the present investigation corrects shortcomings in earlier ones. It is stated that the material energy-tensor is unsatisfactory so that the field-equations of general relativity are taken to be $R_{\mu\nu}=0$, matter being represented by singularities of the field. In manipulating these equations certain combinations of the coefficients of the metric, denoted by γ_{00}, γ_{0n}, γ_{mn}, are employed. The suffix zero denotes the time, whilst m, n $(=1, 2, 3)$ denote spatial, components. The field is assumed to be quasi-static, so that the time-variation of the γ's is small. A method of approximate solution of the field-equations is developed which depends on assuming that, in terms of a parameter λ, power-series expansions of the γ's exist such that: γ_{00} is a series of even powers of λ beginning with λ^2; γ_{0n}, one of odd powers of λ beginning with λ^3; and γ_{mn}, one of even powers of λ beginning with λ^4. These expansions are suggested by what happens when an energy-tensor is used and also by analogy from the theory of half-retarded and half-advanced potentials in electro-magnetic theory. The conditions of integrability of the field-equations, in each successive approximation with respect to powers of λ, are then examined with the aid of surface integrals of the approximations to the tensor $R_{\mu\nu}$ taken around the singularities of the field. Integrability involves the presence of "dipole" solutions, i.e., of solutions one element of which corresponds to negative gravitational masses. These must be rejected on physical grounds and the annihilation of the expressions corresponding to such dipoles produces the equations of motion of the singularities corresponding to particles of positive mass. The results for two particles are obtained in the Newtonian and non-Newtonian approximations and it is also indicated how simplifications can be introduced by the choice of a suitable coordinate-system. *G. C. McVittie* (London).

Clark, G. L. On the gravitational mass of a system of particles. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 412–423 (1949).

In classical mechanics the mass of a system of gravitating particles can be defined to be the mass of an equivalent particle which gives the same field at great distances, or alternatively the mass can be defined by means of Gauss's theorem. Reference to the former procedure was made by Eddington and Clark in a discussion on the problem of n bodies [Proc. Roy. Soc. London. Ser. A. 166, 465–475 (1938)]. The relativistic extension of Gauss's theorem was investigated by E. T. Whittaker for a particular form of the line-element [Proc. Roy. Soc. London. Ser. A. 149, 384–395 (1935)], and for more general fields by Ruse [Proc. Edinburgh Math. Soc. (2) 4, 144–158 (1935)]. The latter, treating the problem geometrically, gave a definition of gravitational force in a general space-time, and expressed the normal component of this force as the sum of two volume integrals. The author of the present paper remarks that the significance of one of these integrals was obvious, but that the meaning of the second was far from clear. He therefore examines the terms in Ruse's result as far as the order m^2 in the case of a fundamental observer at rest, and modifies the Eddington-Clark discussion of 1938 to bring the two investigations into line. He concludes that the surface integral of the normal component of the gravitational force taken over an infinite sphere is -4π times the energy of the system. [The reviewer wishes to note that his definition of gravitational force was not accepted by Synge, Proc. Edinburgh Math. Soc. (2) 5, 93–102 (1937).] *H. S. Ruse.*

Clark, G. L. The equivalence of the gravitational and invariant mass of an isolated body at rest. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 424–426 (1949).

In this appendix to the paper of the preceding review, the author demonstrates the validity of an assumption made in the main paper. The assumption is, in effect, that the gravitational mass of an isolated body at rest is the same as its invariant mass at least as far as the terms in m^2 are concerned. *H. S. Ruse* (Leeds).

Clark, G. L. The internal and external fields of a particle in a gravitational field. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 427–433 (1949).

The author remarks that the gravitational field of a system of particles was investigated by de Sitter in 1916, and that a minor alteration of the analysis was later made by Eddington and himself [Proc. Roy. Soc. London. Ser. A. 166, 465–475 (1938)]. The amended value of the potential g_{44} was the same as that derived by Einstein, Infeld, and Hoffmann without making use of the energy-tensor [Ann. of Math. (2) 39, 65–100 (1938)]. This agreement suggests that the revised de Sitter argument was correct. In the present paper the author shows that this is not the case, since the de Sitter analysis overlooked any possible interaction terms in the stress components of the energy-tensor. He finds the value of these terms, p_{mn}, and shows that the agreement mentioned above is due to the fact that the volume integral of p_{11} vanishes. *H. S. Ruse* (Leeds).

Clark, G. L. The mechanics of continuous matter in the relativity theory. Proc. Roy. Soc. Edinburgh. Sect. A. 62, 434–441 (1949).

The main part of this paper is concerned with the problem of the rotating disc. Having remarked that little progress has hitherto been made in the relativity theory of elasticity, the author gives a brief account of the arguments whereby Lorentz concluded that the radius of a rotating disc, as measured by an observer at rest on the disc, undergoes a contraction. Lorentz's result was accepted by Eddington but attacked by others. In this paper the author argues that both Lorentz and Eddington were considering material in which the waves of dilatation travel with infinite velocity; and that, if one accepts the postulate that the limiting velocity c_0 of these waves is that of light, c, then for "incompressible" matter $(c_0=c)$ there is no alteration of the radius. For $c_0<c$ there is an expansion. The discussion is restricted

throughout to the case of small strain. The paper concludes with an investigation of the equations of equilibrium of a continuous static distribution of matter in the case of weak fields for which the fourth power of the density may be neglected. *H. S. Ruse* (Leeds).

Wyman, Max. Radially symmetric distributions of matter. Physical Rev. (2) **75**, 1930–1936 (1949).

Statical solutions of Einstein's gravitational equations are found corresponding to the interiors of spheres of matter in which the pressure is isotropic, the internal line-element being

$$ds^2=e^{\nu}dt^2-e^{\lambda}dr^2-r^2d\theta^2-r^2\sin^2\theta d\varphi^2,$$

where ν, λ are functions of r to be determined. Each solution must reduce at the outer boundary ($r=r_b$) of a sphere to the exterior Schwarzschild field so that (i) $e^{\nu}=e^{-\lambda}=1-2m/r_b$; (ii) the pressure is zero at $r=r_b$. These boundary conditions together with the Einstein equations are insufficient to solve the problem unless either one of ν, λ, ρ (the density) is given as a function of r, or an equation of state is available. When ρ is constant an exact solution is found in which the relativistic mass m of the sphere is not greater than the Newtonian mass. There is also an upper limit to the size of the sphere for given ρ, and this is larger than the Schwarzschild limit. A general solution involving hypergeometric functions is next obtained by assuming that ρ is proportional to r^{N-2} and here, if $N=4$, the relativistic and Newtonian masses are equal. Lastly, other general solutions are found by assuming in turn: (i) $e^{\nu}=(Ar^{1-n}-Br^{1+n})^2$, where A, B, n are constants; (ii) $e^{-\lambda}=a-(r/R)^N$, where $a=(1+2n-n^2)^{-1}$, $N=2(1+2n-n^2)/(n+1)$ and n, R are arbitrary constants; (iii) $e^{-\lambda}=f(r)e^{\nu}$. *G. C. McVittie* (London).

Barajas, Alberto. On Einstein's principle of equivalence. Comisión Impulsora y Coordinadora de la Investigación Científica (Mexico). Anuario **1945**, 25–30 (1947). (Spanish)

Defining the acceleration of a particle in a general space-time to be d^2x^i/ds^2, the author considers the world-lines, through any given world-point E_0, (x_0^i), of particles acted on solely by gravitational forces. By the principle of equivalence, he says, it must be possible to find a transformation of coordinates $y^i=y^i(x)$ such that the new accelerations d^2y^i/ds^2 are zero at E_0. This leads him to the following conclusions: (1) the world-lines of particles subject only to gravitational forces are the paths of an affine connection B^i_{jk}; (2) the principle of equivalence is valid in Birkhoff's gravitational theory. *H. S. Ruse* (Leeds).

Graef Fernandez, Carlos. The bolometric correction of the magnitudes of the galaxies in Birkhoff's cosmology. Comisión Impulsora y Coordinadora de la Investigación Científica (Mexico). Anuario **1945**, 51–66 (1947). (Spanish)

On the basis of Birkhoff's theory, the author compares the energy received from a real galaxy G' which recedes from an observer with constant velocity v, with that received from an ideal galaxy G at rest relative to the observer, G' and G being intrinsically equal and equidistant from the observer at the instant of emission. He finds that the total energies $E_{G'}$ and E_G received by the observer are in the ratio $E_{G'}/E_G=(1-v)^2/(1+v)^2$ (the velocity of light being unity), and deduces the formula

$$m_G-m_{G'}=5[\log(1-v)-\log(1+v)]$$

for the difference of the bolometric magnitudes of G and G'. *H. S. Ruse* (Leeds).

MECHANICS

Duncan, W. J. Some related oscillation problems. Coll. Aeronaut. Cranfield. Rep. no. **27**, 17 pp. (1949).

A second oscillation problem can be obtained from a first by applying a linear differential operator to the displacements. The second problem has the same frequencies, but different boundary conditions. When these boundary conditions can be physically interpreted, a new problem of interest is solved. Some examples of pairs of systems with related frequency equations are given, which are obtained from the concept of mechanical admittance. *P. Franklin* (Cambridge, Mass.).

Dörr, J. Schwingungen rotierender und rollender Reifen. Ing.-Arch. **15**, 53–64 (1944).

To study the small undamped oscillations for the rim of a vehicular wheel the radial distance as a function of angle is expanded in a Fourier series with coefficients functions of the time. In terms of these coefficients, the system of Lagrange's equations is set up. Eigenfrequencies are calculated, discussed qualitatively, and applied to find critical frequencies in technical applications. *P. Franklin.*

Craig, Homer V. On extensors and the Lagrangian equations of motion. Math. Mag. **22**, 245–251 (1949).

A derivation of Lagrange's equations of motion of a conservative system based on certain assumptions on invariance and dependence on the kinetic and potential energy functions. *P. Franklin* (Cambridge, Mass.).

Castoldi, L. Formulazione lagrangiana delle equazioni del moto per sistemi dinamici soggetti a vincoli servomotori. Nuovo Cimento (9) **6**, 180–186 (1949).

Servomotorial constraints are carefully defined and contrasted with the more usual holonomic smooth constraints. This is done both in theory and with the help of examples. The proof that the Lagrangian equations can be used with systems subject to servomotorial constraints follows trivially from the definition. *D. C. Lewis* (Baltimore, Md.).

Nagabhushanam, K. A generalization of the theorem of virial. J. Indian Math. Soc. (N.S.) **12**, 117–120 (1948).

The first object of this note is to obtain, in terms of the variable (x) of the manifold, the value along the trajectories of $\delta(X_i\dot{x}_i+Ht)$, where X_i is a covariant vector such that the trajectories are the singular curves of its curl, while H is and t is not an integral of motion. Then the familiar form of the virial theorem is obtained by specialising the variables and functions. *M. Haimovici* (Jassy).

***Klouček, C. V. Distribution of Deformation. (A New Method of Structural Analysis).** Translated from the Czech and German editions by A. H. Waddell-Zalud and F. H. Zalud. C. V. Klouček, Prague, 1949. 512 pp. (3 plates).

As might be expected from the title, the method of structural analysis presented in this book is closely related to L. E. Grintner's method of balancing angle changes [Trans.

Amer. Soc. Civil Engrs. **102**, 1020–1036 (1937)]. Whereas Grintner's is an iterative method which follows the general pattern of Southwell relaxation procedure, the present author aims mainly at establishing a method which gives the final result directly. For not too complex frames the new method seems indeed faster than the iterative procedure; as the complexity of the structure increases, however, approximations are introduced which render difficult the comparison between the two methods. *W. Prager.*

Hydrodynamics, Aerodynamics

Théron, Pierre. Sur un théorème d'existence des mouvements des fluides plans avec sillage. C. R. Acad. Sci. Paris **228**, 1922–1923 (1949).

L'auteur étudie le problème déterminé, symétrique du sillage à la Helmholtz, l'obstacle donné C n'étant soumis qu'à des conditions très générales. Soient: O, le point de bifurcation, A le point le plus éloigné de l'axe de symétrie Ox de C. Il existe alors sur OA un point B (pouvant se confondre avec A, exceptionnellement) tel qu'en chaque point $M \epsilon BA$, il se détache un sillage, ne recoupant pas C, ne se recoupant pas et présentant en M une courbure infinie positive (sauf si M coïncide avec B, alors distinct de A). Enfin, les sillages issus de deux points différents de AB ne se recoupent pas.

La méthode de l'auteur est originale; elle offre l'avantage (1) de ne pas s'appliquer seulement aux obstacles tranchants (dont l'intersection avec Ox se réduit à un point unique), (2) de fournir des solutions vérifiant à priori la classique condition de non recoupement de M. Brillouin, (3) d'apporter une contribution au problème dit de la proue.

J. Kravtchenko (Grenoble).

Binnie, A. M. The passage of a perfect fluid through a critical cross-section or "throat." Proc. Roy. Soc. London. Ser. A. **197**, 545–555 (1949).

The steady one-dimensional flows over a broad-crested weir of uniform breadth and through a Venturi flume with a free surface are considered by Hugoniot's method. In both cases, the author finds that the maximum discharge occurs at a critical depth where the velocity of the stream is equal to that of a long wave of small amplitude in the stream. Hugoniot's method is next applied to a swirling liquid flowing under gravity through a vertical nozzle. By the action of the external force in the direction of the flow, the critical condition does not occur at the throat of the nozzle but above it. In an experiment described by the author and Hookings [same Proc. Ser. A. **194**, 398–415 (1948)] the effect appears to be small. Finally, the problem of swirling gas through a nozzle with annular cross-section in the absence of external force is examined, and the throat value of the velocity is evaluated for the two values 2 and 1.4 of the ratio of specific heats. *Y. H. Kuo.*

Synge, J. L. On the motion of three vortices. Canadian J. Math. **1**, 257–270 (1949).

By the configuration of a system of three vortices the author understands the triangle which they form, without regard to rigid body displacements. By the idea of representing a configuration by trilinear coordinates, the author succeeds in elucidating the trajectories of the representative point, and so the characteristics of the changes of configuration, which are found to depend upon the sign of $k_2k_3+k_3k_1+k_1k_2$, where k_1, k_2, k_3 are the strengths of the vortices. The paper contains a number of interesting theorems. *L. M. Milne-Thomson* (Greenwich).

Cârstoiu, I. Sur le mouvement des files de tourbillons alternés indéfinies d'un seul côté. Disquisit. Math. Phys. **6**, 225–233 (1948).

The author studies the motion of the semi-infinite vortex street considered previously by Synge, assuming that the cylinder producing the vortex system suddenly disappears (which the author interprets physically as its stoppage). It is found that the system will not remain stationary, as expected. The author considers the stability at the initial configuration by displacing one vortex. It is found that one of the necessary conditions for stability is the original von Kármán condition, but the present system is always unstable. *C. C. Lin* (Cambridge, Mass.).

Gilbarg, David. A characterization of non-isentropic irrotational flows. Amer. J. Math. **71**, 687–700 (1949).

This paper is unusual in that it attacks a problem of gas dynamics in the large. The objective is to prove that with certain specific exceptions any steady irrotational flow of an inviscid non-piezotropic gas ($p=f(\rho, \eta)$, $f_\rho \neq 0$, $f_\eta \neq 0$) which does not conduct heat is a flow of uniform specific entropy η. This result is often assumed in the engineering literature on the basis of intuitive arguments. From the energy equation it follows at once that the specific entropy of each particle remains constant throughout each region of continuous flow. The appropriate local theorem, given by the reviewer and R. Prim [unpublished report], states that any region of such an irrotational flow, steady or not, is a region of constant entropy or a region where the pressure and density of each particle remain constant, provided the extraneous force is conservative. When the assumptions of steady motion and vanishing extraneous force are added, it follows that the speed is constant upon each stream-line along which the pressure is constant, and hence by a theorem of Hamel [S.-B. Preuss. Akad. Wiss. 1937, 5–20] that any region in which grad $\eta \neq 0$ is a region of helicoidal flow. The author is concerned with proving that with certain specific exceptions an irrotational flow under these assumptions cannot consist of some portions in which grad $\eta = 0$ joined to other portions in which the flow is helicoidal. He proves first that the boundary between two such portions must necessarily be a stream-line belonging to the one or the other, and second that if a stream-line contains one boundary point of region of helicoidal flow, all its points are boundary points, it is itself a circular helix extending to the boundary of the flow, and pressure, density, and speed are constant upon it. Henceforth, he limits his attention to plane and rotationally symmetric flows, and bases his results upon uniqueness theorems applicable to the partial differential equations for the velocity potential in the two cases. These uniqueness theorems concern Cauchy data; for the cases when the flow upon the boundary curve is subsonic or supersonic, known results concerning elliptic and hyperbolic equations respectively are applied, but for the case of a boundary upon which the speed is sonic a new uniqueness theorem for parabolic data valid except when the curve is straight is proved by the author, who conjectures that it holds also for straight boundaries. By applying the uniqueness theorems for the subsonic case the author proves that a plane irrotational flow which is subsonic throughout a region is either a flow of uniform entropy or a

simple vortex throughout the region, and that a rotationally symmetric irrotational flow which is subsonic in a region is either a flow of uniform entropy or a uniform flow throughout. He gives the counter-example of a uniform flow of variable entropy having part of a straight stream-line in common with a uniform flow of constant entropy continued into a corner flow to show that these theorems cannot be extended without exception to the supersonic case. He then defines a non-cusped stream-line as any stream-line which is not an infinite half-line, a circular arc, a closed curve consisting in part of a circular arc with a cusp at each end of the arc, or a curve consisting in part of a half-line with a cusp at the end of a half-line, and proves that the above theorems hold for purely supersonic flows in regions bounded by non-cusped stream-lines. In the case of mixed flows, the theorems so stated would continue to hold if the author's conjectured uniqueness theorem for parabolic data were admitted, but with the somewhat weaker theorem he was able to prove he obtains the result that if a plane irrotational flow in a region bounded by non-cusped stream-lines is not a flow of uniform entropy, and if further the portion of the flow in which grad $\eta \neq 0$ contains a curved stream-line, then the flow is a simple vortex. The author states that if uniqueness theorems stronger than those now known were available for partial differential equations in three independent variables, the results could be extended to flows in three dimensions. *C. Truesdell* (Washington, D. C.).

Miche, Robert. Sur la réduction à un principe variationnel du mouvement non lent des fluides visqueux. J. Math. Pures Appl. (9) **28**, 151–179 (1949).

This study concerns the question of whether it is possible to derive the general equations of motion of an incompressible viscous fluid from a variational principle. The answer is affirmative. To show this, the author first defines a volume integral over a fixed domain D:

$$\Phi = \iiint_D L(\psi^\alpha) d\tau,$$

where L is the Lagrangian. In the case of steady motion, L is a function of vector functions ψ^α and their partial derivatives of order n. Under certain circumstances, this expression may include an extra term $\lambda^\alpha G^\alpha$, and $L+\lambda^\alpha G^\alpha$ is then called the "complete Lagrangian," where λ^α are multipliers and $G^\alpha = 0$ are the equations of motion. The necessary condition for an extremum leads to Euler's equations

$$(1) \qquad E^i(L) = \frac{\partial L}{\partial \psi^\alpha} - \frac{\partial}{\partial x_\alpha}\frac{\partial L}{\partial \psi^i_\alpha} + \frac{\partial^2}{\partial x_\alpha \partial x_\beta}\frac{\partial L}{\partial \psi^i_{\alpha\beta}} - \cdots = 0.$$

Here x_α denotes a vector, while the subscripts of the vector ψ^i denote partial differentiation. If the motion is unsteady, the vector functions ψ^i will depend on x_α as well as the time t. The Lagrangian will contain terms involving the first partial derivatives with respect to t. The condition for an extremum is then subject to three additional equations

$$(2) \qquad E^\alpha(L) = \frac{\partial L}{\partial \psi^i_t} - \frac{\partial}{\partial x_\alpha}\frac{\partial L}{\partial \psi^i_{\alpha t}} + \cdots = 0.$$

As is well known, the problem is essentially indeterminate. In the case of steady motion, there are seven unknown functions ψ^α, λ^α and L for only six equations. On the other hand, if the motion is unsteady, there will be more equations (9) than there are unknowns (7). The author solves this difficulty by choosing the Lagrangian to be the sum of the rate of increase of kinetic energy per unit volume T and that of the dissipation per unit volume 2Ψ. Equations (2) then serve to determine λ^α while equations (1), which correspond to the Navier-Stokes equations, are consequences of the principle of conservation of energy. The solution is verified to be consistent. In the special case of steady motion, $T=0$, and since $\lambda^\alpha G^\alpha = 0$ under all circumstances, $L = 2\Psi$. This constitutes the well-known Helmholtz theorem. Some other special cases are also discussed.

Finally, the author makes the conjecture that his "complete Lagrangian" is a divergence, because in that case Euler's equations will be identically satisfied. This is actually verified in the two-dimensional case. *Y. H. Kuo.*

Riesenkampf, B. Hydraulik der Grundwasserströmungen. IV, V. Abh. Tschernyschewsky Univ. Saratow. **15**, no. 5, 3–93 (1940). (Russian. German summary)

For parts I, II, III see the same Abh. **14**, no. 1, 89–114; no. 2, 181–205 (1938).

Rizenkampf, B. K. On a case of filtration of water in stratified soil. Abh. Tschernyschewsky Univ. Saratow. **15**, no. 5, 94–100 (1940). (Russian)

Ertel, Hans, und Köhler, Hilding. Ein Theorem über die stationäre Wirbelbewegung kompressibler Flüssigkeiten. Z. Angew. Math. Mech. **29**, 109–113 (1949). (German. Russian summary)

Consider steady motion of a barotropic fluid in a rotating coordinate system x, y, z. Suppose $\alpha = \text{constant}$ and $\beta = \text{constant}$ define stream lines and (absolute) vortex lines, respectively. Then the mass-flow vector $\rho\mathbf{v}$ can be expressed as $\rho\mathbf{v} = \epsilon_\alpha(\alpha, H)\ \text{grad}\ \alpha \times \text{grad}\ H$ and the absolute vorticity vector as $\mathbf{w} = \epsilon_\beta(\beta, H)\ \text{grad}\ \beta \times \text{grad}\ H$, where H denotes the stagnation enthalpy and ϵ_α, ϵ_β are functions to be determined by the equations $\text{div}\ \rho\mathbf{v} = 0 = \text{div}\ \mathbf{w}$. But Bjerknes' theorem $\mathbf{v} \times \mathbf{w} = \text{grad}\ H$ now leads to a condition involving the three functions α, β, H:

$$\epsilon_\alpha \epsilon_\beta \partial(\alpha, \beta, H)/\partial(x, y, z) = \rho,$$

where ρ is the density. From this relation the author is able to deduce the following theorem concerning the (α, H)-tubes and (β, H)-tubes into which the space is divided by intersecting surfaces $\alpha = \text{constant}$, $\beta = \text{constant}$, and $H = \text{constant}$, and the (α, β, H)-cells that are formed by the intersection of these tubes: the product of the mass flux through an (α, H)-tube and the vortex strength of a (β, H)-tube is numerically equal to the mass of the (α, β, H)-cell. *W. R. Sears* (Ithaca, N. Y.).

Moreau, Jean-Jacques. Sur l'interprétation tourbillonnaire des surfaces de glissement. C. R. Acad. Sci. Paris **228**, 1923–1925 (1949).

The author's objective is to determine the change of the surface curl upon a vortex-sheet as a motion proceeds. He states a kinematical formula for its derivative relative to a point fixed with respect to a surface coinciding with the vortex sheet and endowed with a velocity which is the mean of those upon its two sides. From this formula, he concludes that in barotropic motions of an inviscid fluid, subject to conservative extraneous force, there are theorems for the surface curl which are direct analogues of those of Helmholtz and Kelvin for continuous motions. [Since the author employs the false assumption that for such motions there is no tangential discontinuity in the acceleration (one of many counter-examples: $u = kxt$, $v = 0$, $w = 0$, $\rho = \text{constant}$,

Please see Errata p. 871

extraneous force $=0$, $p=-\rho k(1+kt)x^2/2$, $k=k(y)=1$ if $y>0$, $=0$ if $y<0$), the reviewer feels some hesitance in accepting the result.] *C. Truesdell* (Washington, D. C.).

v. Weizsäcker, C. F. Das Spektrum der Turbulenz bei grossen Reynoldsschen Zahlen. Z. Physik **124**, 614–627 (1948).

The author considers a volume filled with isotropic turbulence of linear dimension L_0, subdivided into successively smaller volumes of linear dimensions L_n, and discusses the mean value of the physical quantities over these volumes. The condition of equilibrium of energy dissipation leads to the following law: the average value of the magnitude of relative velocity for two points at a distance L_n apart is proportional to $L_n^{\frac{1}{3}}$. The turbulence exchange over a distance L_n is then proportional to $L_n^{\frac{4}{3}}$. In terms of the Fourier analysis of the flow field, this means that the spectrum is proportional to the negative five-thirds power of the wave number. *C. C. Lin* (Cambridge, Mass.).

Heisenberg, W. Zur statistischen Theorie der Turbulenz. Z. Physik **124**, 628–657 (1948).

This paper develops the statistical theory of isotropic turbulence from the point of view of spectrum analysis. The author develops the relation between the spectra obtained from a one-dimensional Fourier analysis and from a three-dimensional analysis. An assumption is introduced regarding the transfer mechanism, leading to a definite law of spectrum for the range of frequencies chiefly responsible for dissipation. [In this paper, the author uses an interpolation formula; but the exact law has been obtained later by Chandrasekhar, Physical Rev. (2) **75**, 896–897 (1949); these Rev. **10**, 493.] The mean pressure fluctuation and the velocity correlation function are calculated, and the constant of proportionality appearing in the transfer function is estimated. [It should be noted that the above two papers were submitted in 1946. In the meantime, developments based on these ideas have already been published.] *C. C. Lin* (Cambridge, Mass.).

Heisenberg, W. On the theory of statistical and isotropic turbulence. Proc. Roy. Soc. London. Ser. A. **195**, 402–406 (1948).

The author studies the spectrum with the help of his assumption for the transfer of energy. A similarity hypothesis is introduced. Certain approximate solutions for the shape of the spectrum are given. It is concluded that excepting for the extreme low frequency range where the spectrum behaves like the fourth power of the wave number [see the following review], the spectrum behaves successively as the first power, the negative 5/3 power and the negative seventh power. It is pointed out that the spectrum will remain essentially similar when the linear part of the spectrum is present. It is concluded that "turbulence is an essentially statistical problem of the same type as one meets in statistical mechanics, since it is the problem of distribution of energy among a very large number of degrees of freedom. Just as, in Maxwell theory, this problem can be solved without going into any details of the mechanical motion, so it can be solved here by simple considerations of similarity." *C. C. Lin* (Cambridge, Mass.).

Batchelor, G. K. The role of big eddies in homogeneous turbulence. Proc. Roy. Soc. London. Ser. A. **195**, 513–532 (1949).

The spectrum tensor, defined as the three-dimensional Fourier transform of the double velocity correlation tensor, is introduced for homogeneous turbulence, its properties are studied, and the energy equation discussed by deriving it from a Fourier transform of the Kármán-Howarth equation for the double correlation tensor. It is shown that, for small values of the wave number k, the energy per unit interval of wave-number magnitude varies as k^4, and that the coefficient of proportionality is a constant. [These results are also contained in the reviewer's paper reviewed below.] The final stage of decay is discussed. Similarity hypotheses and the role of big eddies in limiting them are discussed. The author refers to a fuller discussion by Heisenberg [see the preceding review]. *C. C. Lin* (Cambridge, Mass.).

***Lin, C. C. Remarks on the spectrum of turbulence.** Proc. Symposia Appl. Math., Vol. I, pp. 81–86. American Mathematical Society, New York, N. Y., 1949. $5.25.

Since von Kármán and Howarth [Proc. Roy. Soc. London. Ser. A. **164**, 192–215 (1938)] demonstrated that the propagation of the double correlation function $f(r)$ in isotropic turbulence is governed by the triple correlation $h(r)$, much work has been directed toward investigating the spectral function $F_1(r)$ which G. I. Taylor [Proc. Roy. Soc. London, Ser. A. **164**, 476–490 (1938)] had showed was a Fourier transform of the double correlation. In the present paper it is demonstrated that the time variation of the spectral function $F_1(k)$ is governed by a relation obtained from the Kármán-Howarth equation by a three-dimensional Fourier transform. The appropriate spectral function is found to differ from that defined by Taylor, but is related to it by $F(k)=k^2F_1''(k)-kF_1'(k)$ which is equivalent to the form given by Heisenberg [see the third preceding review]. Furthermore the equation giving the change of spectrum is

$$(\partial/\partial t)\{\bar{u}^2F(k)\}+2(\bar{u}^2)^{\frac{1}{2}}k^2H(k)=-2\nu\bar{u}^2k^2F(k),$$

where $H(k)=2\{k^2H_1''(k)-kH_1'(k)\}$ and $H_1(k)$ is the sine transform of the triple correlation function $h(r)$: $H_1(k)=2\pi^{-1}k^{-1}\int_0^\infty h(r)\sin kr\,dr$. Both Obukhoff [C. R. (Doklady) Acad. Sci. URSS (N.S.) **32**, 19–21 (1941); these Rev. **3**, 221] and Heisenberg [loc. cit.] have assumed forms for the equation of spectrum variation which tacitly involve the assumption of an integral relation between the function $H(k)$ and the spectral function. The author discusses these relations in the light of the rigorous spectral equation.

From the spectral equation and the known even behavior of both $F_1(k)$ and $H_1(k)$ for low values of k, the author demonstrates simply that the spectrum is unchanged at low wave numbers and hence the integral $\int_0^\infty f(r)r^4\,dr$ is time invariant. This is equivalent to and gives another interpretation of the Loitzianski invariant. The relationship of the turbulence spectrum to the time distribution of zero fluctuating velocity is discussed briefly.

F. E. Marble (Pasadena, Calif.).

Tempest, R. K., and Rosenhead, L. Notes on the linearised equation for the velocity potential of the steady supersonic flow of a compressible fluid. Proc. London Math. Soc. (2) **51**, 197–212 (1949).

Various general solutions of the two-dimensional wave equation have been collected and interpreted with reference to the linearized equation for steady supersonic potential flow, $\partial^2\phi/\partial x^2+\partial^2\phi/\partial y^2=\alpha^2\partial^2\phi/\partial z^2$, where $\alpha^2=M^2-1>0$. They include, as special cases, linearized flows considered by numerous authors. For example, the Kármán-Moore solution for a cone [Trans. A.S.M.E. **54**, Applied Mechanics

303–310 (1932)] is both a Legendre function and a Whittaker integral solution. In cylindrical or rectangular coordinates the search for homogeneous solutions $\phi = z^n\Phi(t)$, where $t = z/r\alpha$ or $z/y\alpha$, leads to expressions for $\Phi(t)$ in terms of hypergeometric functions. Some of these axisymmetric "cone" solutions were previously given by Hayes [Quart. Appl. Math. 4, 255–261 (1946); these Rev. 8, 237]. By means of Gauss's relation for hypergeometric functions most of the plane "wedge" solutions can be transformed into the natural forms $A(z \pm \alpha y)^n$ for some constant A.

J. H. Giese (Havre de Grace, Md.).

Bailey, W. N. A note on the paper by Tempest and Rosenhead. Proc. London Math. Soc. (2) 51, 213–214 (1949).

By means of Kummer's formula for hypergeometric functions the rest of the plane solutions $z^n\Phi(z/y\alpha)$ found in the preceding paper have been transformed into the form $A[(z+\alpha y)^n \pm (\pm 1)^n(z-\alpha y)^n]$. *J. H. Giese.*

Wendt, H. Die Jansen-Rayleighsche Näherung zur Berechnung von Unterschallströmungen. S.-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. 1948, no. 7, 26 pp. (1948).

The well-known method of approximation by expansion in powers of M^2, where M denotes the stream Mach number, is treated by use of complex variables. As an example, the symmetrical flow past a circular cylinder between parallel channel walls is calculated in the first approximation, i.e., including the terms in M^2. *W. R. Sears* (Ithaca, N. Y.).

Lin, C. C. On the subsonic flow through circular and straight lattices of airfoils. J. Math. Physics 28, 117–130 (1949).

In this paper the subsonic compressible flow through infinite straight lattices and through circular arrangements of airfoils (as in a vaned diffuser) is considered within the approximation of the von Kármán-Tsien method (linear pressure-volume relation) as extended by Gelbart [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1170 (1947); these Rev. 8, 417] and Lin [Quart. Appl. Math. 4, 291–297 (1946); these Rev. 8, 418]. After a brief discussion of the nature of such flows, their mapping into flow around a circle is undertaken. The circular lattice is treated in detail and the straight lattice obtained as a special case. In the closing discussion the author points out that the method is not essentially related to the hodograph plane, although the historical development has been that way. In the present paper only the basic theory is worked out, without detailed application or the determination of lattice characteristics.

W. R. Sears (Ithaca, N. Y.).

Holt, M. Flow patterns and the method of characteristics near a sonic line. Quart. J. Mech. Appl. Math. 2, 246–256 (1949).

When initial data are given on a sonic line, the usual method of characteristics cannot be used to extend the flow into the supersonic region. The author presents two methods that may be used. The first is based on the double Taylor series for the velocity components near a sonic point. The second makes use of the fact that the coordinates of points and the values of flow quantities on a Mach line can be expressed as functions of the variable ν, which defines the local Mach number. Thus the coordinates x and y of the Mach lines through a sonic point are expressed in power series in ν, the coefficients being functions of the velocity derivatives at the sonic point. For a "singular sonic point," i.e., one at which the streamline curvature vanishes, a different technique is required. It is shown how these methods may be used to continue a supersonic field right up to a sonic line. The extension to axi-symmetric flows is sketched. *W. R. Sears* (Ithaca, N. Y.).

Ward, G. N. Supersonic flow past thin wings. I. General theory. Quart. J. Mech. Appl. Math. 2, 136–152 (1949).

The author's summary is as follows. The linearized potential problem of supersonic flow is solved for nearly plane wings of arbitrary shape of planform and section. The velocity field is divided into its symmetrical and antisymmetrical parts, which are treated separately. The results for the symmetrical part have been given before, by A. E. Puckett [J. Aeronaut. Sci. 13, 475–484 (1946); these Rev. 8, 109], and are only considered briefly. The results for the antisymmetrical part are determined by extending an integral equation method used previously by J. C. Evvard [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1382 (1947); these Rev. 8, 610] in a more restricted problem. The Kutta-Joukowski condition is applied to the velocity at the trailing edge in order to make the solution determinate. Expressions are given for the potential on the wing, from which the velocities can be obtained. For the smaller aspect ratios, the analysis becomes complicated and only an indication of how the results can be obtained is given. The velocity is singular at the subsonic leading edges and the forces on the wing have to be determined by a limiting process which is illustrated by obtaining expressions for the lift and drag forces. All the results are completely general and no special cases are considered. *W. R. Sears* (Ithaca, N. Y.).

Miles, John W. Quasi-stationary thin airfoil theory. J. Aeronaut. Sci. 16, 440 (1949).

The author writes approximate formulas for lift and moment coefficients of a two-dimensional thin airfoil performing steady oscillations in an incompressible fluid. These are obtained from well-known results of unsteady airfoil theory but only first-order (in reduced frequency k) terms are retained. For plunging motion (oscillations in vertical translation), the so-called quasi-steady formulas, which are taken from steady airfoil theory and simply neglect instationary effects, are found to be correct to order $k^2 \log k$. For pitching motions about the quarter-chord, however, errors of order $k \log k$ and k are committed when the quasi-steady formulas are used. The note ends with a discussion of the practical significance of this, and a comment on compressibility corrections. *W. R. Sears* (Ithaca, N. Y.).

Weissinger, Johannes. Über eine Erweiterung der Prandtlschen Theorie der tragenden Linie. Math. Nachr. 2, 45–106 (1949).

[This is a detailed report on the author's work which was first reported in Germany in 1942 and has since become familiar to engineers in Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1120 (1947); these Rev. 8, 542.] For a thin lifting surface in a uniform stream of an ideal incompressible fluid, an integro-differential equation can be written for the bound-vortex strength distribution over the surface, satisfying the given slope conditions at points of the surface. This equation is put into a form where it appears as the well-known Prandtl lifting-line equation with a correction term in the induced velocity. This correction term is now calculated by an approximate method. This is the author's "F (Fläche)-method"; it amounts to determining the whole induction effect by means of the induced downwash at the

¼-chord line, this downwash being calculated for a standard chordwise, but arbitrary spanwise, bound-vortex distribution. A numerical method for routine use of this procedure is worked out, but because it requires excessive labor, a further simplification is made in the "L (Linie)-method." Here the idea of a chordwise distribution of vortices is discarded, and the bound- and trailing-vortex effects are evaluated for a single concentrated lifting line at the ¼-chord line. Again these effects are evaluated at the ¾-chord points to determine their effect on circulation. This method is no more difficult to employ than the customary procedures developed for the Prandtl theory, provided certain numerical tables are at hand; it has the great advantage of being able to handle yawed and sweptback wings.

The author's numerical results show the L-method to be a close approximation to the F-method in practical cases. In a footnote it is mentioned that the L-method is the same as Mutterperl's [Tech. Notes Nat. Adv. Comm. Aeronaut., no. 834 (1941)], except for the numerical treatment. Tables of numerical constants for use with the F- and L-methods are given. In the numerical formulation, use is made of a numerical-quadrature formula of Multhopp [Luftfahrtforschung 15, 153–169 (1938)]. In a mathematical supplement the author considers the extension of his methods to various classes of planforms, the properties of certain functions, and other details. [For another recent paper on this subject, see E. Reissner, Proc. Nat. Acad. Sci. U. S. A. 35, 208–215 (1949); these Rev. 10, 753.] *W. R. Sears.*

Garrick, I. E., and Rubinow, S. I. Flutter and oscillating air-force calculations for an airfoil in a two-dimensional supersonic flow. Tech. Rep. Nat. Adv. Comm. Aeronaut., no. 846, 25 pp. (1946).

The authors' summary is as follows. A connected account is given of the Possio theory of non-stationary flow for small disturbances in a two-dimensional supersonic flow and of its application to the determination of the aerodynamic forces on an oscillating airfoil. Further application is made to the problem of wing flutter in the degrees of freedom: torsion, bending, and aileron rotation. Numerical tables for flutter calculations are provided for various values of the Mach number greater than unity. Results for bending-torsion wing flutter are shown in figures and are discussed. The static instabilities of divergence and aileron reversal are examined as is a one-degree-of-freedom case of torsional oscillatory instability. *D. Gilbarg* (Bloomington, Ind.).

***Tables of Supersonic Flow Around Cones of Large Yaw, by the Staff of the Computing Section, Center of Analysis, Under the Direction of Zdeněk Kopal.** Massachusetts Institute of Technology, Department of Electrical Engineering, Center of Analysis. Technical Report No. 5. Cambridge, Mass., 1949. xviii+125 pp. (1 plate)

These tables are based on unpublished calculations by A. H. Stone. In previous publications [same reports, no. 3 (1947); also Stone, J. Math. Physics 27, 67–81 (1948); these Rev. 9, 112, 544], the flow about a cone at a small angle of yaw ϵ was calculated by expanding in powers of ϵ and discarding $O(\epsilon^3)$. In his new work, Stone has evaluated the ϵ^2 terms, so that a second approximation is possible. Only the results used in numerical computations are reproduced here. It is found that the shock wave is, to the second approximation, conical, and is yawed to the same angle as predicted by the first-order theory. It is, however, not a circular cone, but is nearly elliptic. The head drag, which was unaffected by yaw in the first approximation, involves second-order corrections, but the normal force remains the same as in the first approximation. The tables permit all the interesting flow quantities for the region between the cone and the shock wave to be calculated, to the second order, for seven cones having semi-apex angles between 5° and 25°. In each case a range of Mach numbers is covered. *W. R. Sears.*

Iglisch, Rudolf. Exact calculation of laminar boundary layer in longitudinal flow over a flat plate with homogeneous suction. Tech. Memos. Nat. Adv. Comm. Aeronaut., no. 1205, 69 pp. (1949).

[Translated from Schriften der Deutschen Akademie der Luftfahrtforschung 8B, no. 1 (1944).] The problem considered is the numerical calculation of the incompressible laminar boundary layer on a semi-infinite flat plate with suction; the particular case of homogeneous suction is treated in detail. If u and v are the fluid velocity components in the x and y directions respectively, the mathematical problem consists of the standard boundary layer equations $uu_x+vu_y=\nu u_{yy}$, $u_x+v_y=0$, together with the conditions $u(x,0)=0$, $u(x,\infty)=U_0$, $v(x,0)=v_0(x)$, $u(0,y)=U_0$, where $v_0(x)$ denotes the distribution of suction or exhaust velocity at the surface. By introducing the stream function $\psi(x,y)$ and carrying out a succession of coordinate transformations, the problem reduces to the solution of the nonlinear parabolic equation

$$V^{\frac{1}{2}}V_{\tau\tau}+V_\tau\cdot(2\tau\sigma+\tau^3-\tau^{-1}V^{\frac{1}{2}})=2\tau^3\sigma V_\sigma$$

with conditions $V(\infty,0)=4$, $V(0,\sigma)=0$, where $V(\tau,\sigma)U_0^2/4$ is the square of the velocity parallel to the plate. The new independent variables are, in the case of homogeneous suction,

$$\sigma=-v_0\{2x/(\nu U_0)\}^{\frac{1}{2}},$$

$$\tau=\{[\psi(x,y)+v_0(2x/(\nu U_0))]/[2x/(\nu U_0)]^{\frac{1}{2}}\}^{\frac{1}{2}}.$$

To obtain solutions, the fundamental differential equation is treated as a difference equation in σ; if the solution for a constant value σ_1 is known, the solution for another constant value $\sigma_1+\Delta\sigma$ is found by considering the ordinary differential equation in τ,

$$\bar{V}^{\frac{1}{2}}V''+(2\tau\sigma+\tau^3-\tau^{-1}\bar{V}^{\frac{1}{2}})V'=2\tau^3\sigma(V-\bar{V})/\Delta\sigma$$

with $V(0)=0$, $V(\infty)=4$, where $\bar{V}$ represents the known solution for $\sigma=\sigma_1$. The author solves this equation by numerical evaluation of the coefficients in the series $[V(\tau)]^{\frac{1}{2}}=\sum_{n=1}^{\infty}c_n\tau^n$. The starting point of the numerical analysis is taken to be the known Blasius solution for which $\sigma=0$, i.e., zero suction. The distribution of velocity parallel to the plate together with the first and second derivatives taken normal to the plate are tabulated for $0\leqq\sigma\leqq4.0$. Corresponding values of the displacement and momentum thicknesses are also given. *F. E. Marble.*

Weinstein, Alexander. On surface waves. Canadian J. Math. 1, 271–278 (1949).

The author shows that the mixed boundary value problem associated with linearized surface wave equations can be conveniently solved by a combination of the method of substitution of the dependent variable to reduce the problem to one with the classical boundary condition, and the method of separation of variables. *H. S. Tsien.*

Ursell, F. On the heaving motion of a circular cylinder on the surface of a fluid. Quart. J. Mech. Appl. Math. 2, 218–231 (1949).

The author considers the wave motion generated by forced vertical oscillation of a circular cylinder with the

mean position of its center at the undisturbed water surface. The analysis is based on the assumption that both waves and oscillation are of small amplitude. Expressions are obtained for the ratio of the wave amplitude at infinity to the amplitude of the oscillation and for the increase in inertia (virtual mass) caused by the motion of the water. The results of computation of these two quantities are tabulated for several values of $\sigma^2 a/g\pi$ between 0 and $\frac{3}{4}$, where σ is the frequency of oscillation, a is the radius of the cylinder and g is the acceleration due to gravity. *J. V. Wehausen.*

Borodin, V. A., and Dityakin, Yu. F. Unstable capillary waves on the surface of separation of two viscous fluids. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 267–276 (1949). (Russian)

Rayleigh a le premier fait l'étude théorique de la décomposition d'un jet liquide et a trouvé les conditions pour la formation des gouttes de même diamètre. D'autres auteurs ont amélioré la théorie en introduisant la viscosité et la densité du milieu environnant et ont obtenu des résultats en bonne concordance avec les expériences. Dans le présent mémoire les auteurs étudient pour la première fois la possibilité de formation des gouttes de différents diamètres en étudiant les ondes capillaires instables à la surface de séparation de deux fluides visqueux.

Pour simplifier le problème et permettre l'étude algébrique de l'équation caractéristique les auteurs supposent que le rayon du jet est infiniment grand par rapport aux longueurs des ondes capillaires à la surface du jet liquide. Evidemment cette hypothèse est trop grossière dans l'étude d'un phénomène aussi complexe, mais grâce à elle on peut avoir une idée qualitative du mode de formation des gouttes de différents diamètres. En supposant les deux fluides non pesants et incompressibles et en négligeant les infiniment petits de second ordre des vitesses perturbées ainsi que de leurs dérivées jusqu'au troisième ordre, les auteurs obtiennent en utilisant l'équation de Navier-Stokes et l'équation de continuité, deux équations aux dérivées partielles linéarisées de la forme de Helmholtz, l'une pour le premier fluide, l'autre pour le second.

En introduisant les conditions aux limites et en supposant les fluides immobiles et ayant le même coefficient de viscosité on obtient une équation caractéristique qui est un polynome de huitième degré dont les coefficients dépendent de deux paramètres. L'étude de cette équation pour les valeurs positives des paramètres permet de démontrer l'existence à la surface de séparation de trois ondes instables.

M. Kiveliovitch (Paris).

Dvuhšerstov, G. I. Hydraulic shock in tubes of non-circular section and in the flow of a fluid between elastic walls. Učenye Zapiski Moskov. Gos. Univ. Mehanika 122, tom II, 17–76 (1948). (Russian)

The purpose of this paper is to investigate the effect of the elasticity of the walls and the shape of the cross-section upon the pressure behind and the speed of propagation of the shock wave caused by the sudden closing of a valve in a pipe or conduit. The theory of shells is applied to determine the deformation of the cross-section resulting from the hydraulic shock pressures. Inertia of the walls is neglected. Shapes considered are rectangles, regular polygons and ovals of small non-circularity. The reduction of peak pressure resulting from even quite small non-circularity is very substantial. The effect of a short length of pipe of non-circular section placed between the valve and a long section of circular pipe is also discussed and apparently, when properly chosen, may also cause a substantial reduction.

J. V. Wehausen (Falls Church, Va.).

Calder, K. L. Eddy diffusion and evaporation in flow over aerodynamically smooth and rough surfaces: a treatment based on laboratory laws of turbulent flow with special reference to conditions in the lower atmosphere. Quart. J. Mech. Appl. Math. 2, 153–176 (1949).

The problem mentioned in the title is attacked assuming that the wind velocity u and the vertical component of eddy viscosity $K(z)$ may, in certain ranges, be considered as varying as powers of the altitude z above the surface. The first main problem considered is that of diffusion of matter from an infinite line source lying on the ground, oriented perpendicular to the direction of the wind. The density χ of diffused matter at a distance x downwind and an altitude z is taken as satisfying the equation

$$u(\partial\chi/\partial x) = (\partial/\partial z)[K(z)(\partial\chi/\partial z)].$$

A solution of this, infinite at the source, vanishing at infinity, and satisfying the boundary condition $K(\partial\chi/\partial z)=0$ when $z=0$, is given as $\chi = Ax^{-(a+1)/(3a+1)} \exp[-kz^{3a+1}/x]$, where a, A, k are constants determined empirically, but in different ways according as the ground is smooth or rough. A theory of evaporation from an area of finite extent is developed by integrating sources of this sort over an interval $0 \leqq x \leqq x_0$. Assuming a constant concentration, as $z \to 0$, in this interval, the distribution of source strength is obtained as a solution to Abel's integral equation. The vapor concentration may be expressed in terms of incomplete gamma functions at points over the source strip, and in terms of definite integrals amenable to numerical integration elsewhere. Experimental data and numerical calculations given agree quite well.

E. Pinney (Berkeley, Calif.).

Obuhov, A. M. The local structure of atmospheric turbulence. Doklady Akad. Nauk SSSR (N.S.) 67, 643–646 (1949). (Russian)

James, R. W. Translation and development in two-dimensional fields, with special reference to pressure variations. Australian J. Sci. Research. Ser. A. 1, 412–422 (1948).

Equations are developed for the speed of horizontal translation of configurations of scalar quantities. The formulae differ from those of S. Petterssen in that they give average speeds over finite areas rather than speeds at specific points. It is shown that regions of large gradients of the scalars are to be given the greatest weight in order that the average speeds be the "best" by the criterion of least squares. The results further lead to formulae for computation of the changes of the configurations. *H. Panofsky.*

Sekerž-Zen'kovič, Ya. I. On the influence of the form of the shore line on the strength of the monsoon field. Izvestiya Akad. Nauk SSSR. Ser. Geograf. Geofiz. 13, 194–211 (1949). (Russian)

Šuleikin is said to have set up for the so-called temperature anomaly $\tau(x, y)$ caused in winter by the transfer of accumulated heat from the sea to the atmosphere the following differential equations:

$$(1) \qquad \frac{\partial^2\tau}{\partial x^2} + \frac{\partial^2\tau}{\partial y^2} - x_1^2\tau + b^2 = 0,$$

$$(2) \qquad \frac{\partial^2\tau}{\partial x^2} + \frac{\partial^2\tau}{\partial y^2} - x_2^2\tau = 0,$$

where $x_1^2=(\sigma+\alpha)/(AH)$, $x_2^2=\sigma/(AH)$, $b^2=\alpha\tau_w/(AH)$. Here σ denotes a coefficient which characterizes the dissipation of heat into interplanetary space; α, a coefficient characteristic for the transfer of heat from sea to atmosphere; τ_w, the free surface temperature of the water which is supposed to be constant; A, a coefficient connected with the turbulence of the air and H the height of the active monsoon layer. The first equation must be satisfied over the sea, the second over the continent. The boundary-value problem consists here of the equations (1) and (2) and the boundary condition which assigns a constant value to the function $\tau(x, y)$ along the shore line of the island. The author gives the solution in the cases of an elliptic island and a parabolic peninsula. It is shown that against an acute cape the strength of a monsoon field increases greatly.

E. Leimanis (Vancouver, B. C.).

Elasticity, Plasticity

***Murnaghan, F. D. The foundations of the theory of elasticity.** Proc. Symposia Appl. Math., Vol. I, pp. 158–174. American Mathematical Society, New York, N. Y., 1949. $5.25.

The paper gives a general formulation of the theory of finite strain, valid under conditions such that plastic deformation does not occur (e.g., compression under hydrostatic pressure). Matrix notation is used, which makes the formulae very compact. Assuming the existence of a free energy function in isothermal change, the author obtains the following formula for the stress T resulting from a finite displacement which moves a particle from $a=(a^1, a^2, a^3)$ to $x=(x^1, x^2, x^3)$: $T=\rho(x)J^*(\partial\phi/\partial\epsilon)J$. Here J is the Jacobian matrix $\partial x/\partial a$, J^* its transpose, $\rho(x)$ the final density, so that (by conservation of mass) $\rho(x)=\rho(a)\,|\det J|^{-1}$, ϕ is the free energy per unit mass, assumed a function of the strain $\epsilon=\frac{1}{2}(J^*J-E)$, $E=$unit matrix. Thus T is a function of the nine elements of $\partial x/\partial a$, but the author emphasises that it is not, in general, a function of the strain ϵ only. The above formula for T appears to be the central point of the paper. From it specialisations are made. Thus, for an isotropic body, ϕ is a function of the three invariants under rotation of the matrix ϵ. The case of hydrostatic pressure is considered, with $T=-pE$, $\epsilon=-eE$ (p, e scalars), and the following formula obtained: $p=A[(v_0/v)^a-1]$, where v_0 is the value of the volume v when $p=0$, and A, a are constants characterising the medium. The author claims good agreement between this formula and the experimental results of Bridgman for the compression of indium. The uniform tension of an elastic cylinder is also discussed.

J. L. Synge (Dublin).

Picone, M. Esistenza e calcolo della soluzione di un certo problema al contorno per il sistema di equazioni dell'elasticità. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 3, 427–435 (1947).

The author gives a general procedure for integrating the equations of the infinitesimal theory of elasticity. For the sake of brevity, only a formal outline is presented. The author derives a system of six equations for the stress components in terms of the rotation vector W_i. Three are of the type $t_{xx,y}-t_{xy,x}=2\mu W_{x,z}$, while three are of the type $\nabla^2 t_{yz}=\lambda(f_{y,z}-f_{z,y})/(\lambda+2\mu)-4\mu(\lambda+\mu)(W_{x,xx}-W_{z,xz})/(\lambda+2\mu)$. If W_i is a solenoidal field satisfying the usual rotation equations $2\mu\nabla^2 W_x+f_{z,y}-f_{y,z}=0$, etc., then the shearing stresses may be found by integrating the author's equations. A further integration yields the normal stresses, and a still further integration yields the displacements. The author is able to conclude that if a body of any connectivity is subject to a conservative force field whose potential is of the form $L(x)+M(y)+N(z)$, then if the rotations and shearing stresses are constant upon the boundary they are constant throughout, and the normal forces and displacements may be exhibited by simple formulae. For the case of extraneous forces which are polynomials the author writes down a formula giving all the polynomial solutions of the general equations. He then states a new uniqueness theorem, too long to repeat here. The boundary conditions to be satisfied are quite unusual; the displacements and normal stresses are prescribed at two interior points, one shearing stress and two first derivatives of the rotation components are prescribed on the boundary, the rotation components themselves are prescribed on three curves on the boundary, and two shearing stresses are prescribed on two curves on the boundary. For the case of a rectangular parallelepiped, the solution is exhibited in terms of double and triple integrals of doubly infinite series.

C. Truesdell.

Picone, Mauro. Esistenza e calcolo della soluzione di un certo problema al contorno per il sistema di equazioni dell'elasticità bidimensionale. Boll. Un. Mat. Ital. (3) 3, 4–6 (1948).

This note states the simplifications resulting when the author's general method of integrating the equations of elasticity [see the preceding review] is applied to a plane system.

C. Truesdell (Washington, D. C.).

Picone, Mauro. Sulla torsione di un prisma elastico cavo secondo la teoria di Saint Venant. Ann. Soc. Polon. Math. 20 (1947), 347–372 (1948).

The author is concerned with the existence and uniqueness and method of calculating a solution of the torsion problem for the rectangular annulus cross-section whose vertices are the points $(\pm p_0, \pm q_0)$, $(\pm p, \pm q)$, where $p_0<p$, $q_0<q$. He calls a function $g(x, y)$ "uniformly summable" if corresponding to any ϵ there exists a $\delta(\epsilon)$ such that $\int|g(x, y)|\,ds<\epsilon$ when the path of integration is any straight line segment contained in the interior of the cross-section and lying upon a coordinate line and of length inferior to $\delta(\epsilon)$, s being arc length along the line segment. Let H denote the boundaries $x=$constant, except for the corner points, and K the boundaries $y=$constant, except for the corner points. The author calls a function ϕ a solution of the torsion problem for the region if (a) it is harmonic in the interior R, (b) ϕ_x is continuous upon $R+H$, and ϕ_y is continuous upon $R+K$, (c) $\phi_x=y$ upon H and $\phi_y=-x$ upon K. He then proves that if one uniformly summable solution exists, there exists one and only one which is an odd function of x and an odd function of y, and all other uniformly summable solutions differ from this one by an arbitrary constant. The existence of such a solution ϕ is then demonstrated and it is shown that its domain of regularity includes all the closed annulus except possibly the eight vertices, that ϕ_x and ϕ_y are continuous at the four inner vertices, and that at the outer vertices they can become infinite no more rapidly than the $-\frac{1}{3}$ power of the distance. The approximative procedure suggested by the proof of this theorem does not seem to be practicable numerically so the author gives a second existence proof of

a wholly different type. The statement of the result is too long to be given here but the essence of the author's method is to reduce the torsion problem to a novel Dirichlet problem, in which an even function ω harmonic in R and satisfying $\omega=1$ on H and $\omega=-1$ on K is sought, uniqueness being secured by the requirement that certain combinations of line integrals of ω, ω_x, and ω_y along the boundaries shall vanish, while certain other integrals shall be uniformly summable. The author proves that such a harmonic function exists and is related to the torsion function ϕ by the equation $\phi_{xy}=\omega$. Thus if ω can be constructed by an approximative procedure, approximate solutions ϕ_x and ϕ_y of the torsion problem may be obtained by two quadratures.

C. Truesdell (Washington, D. C.).

Džanelidze, G. Yu. The generalized relations in the theory of thin rods. Doklady Akad. Nauk SSSR (N.S.) **66**, 597–600 (1949). (Russian)

The note contains a discussion of the effect produced by generalizing Kirchhoff's approximate formulas of the theory of thin rods [see Love, A Treatise on the Mathematical Theory of Elasticity, 4th ed., Cambridge University Press, 1927, section 255] by including in the expression for the twisting moment the term depending on the second derivative of the angle of twist. *I. S. Sokolnikoff.*

Mitra, D. N. Torsion and flexure of an isotropic elastic cylinder whose cross-section is a semi-cardiod. Bull. Calcutta Math. Soc. **41**, 28–40 (1949).

This paper provides a complete solution of the torsion and flexure problem of a prism whose cross section is a semi-cardioid. It employs the methods of S. Ghosh [same Bull. **39**, 107–112 (1947); **40**, 77–82 (1948); these Rev. **10**, 84, 340]. Explicit results are given for the torsional rigidity and for the coordinates of the center of flexure.

D. L. Holl (Ames, Iowa).

Birman, S. E. On the problem of the torsion of a hollow rod with a square cross section. Doklady Akad. Nauk SSSR (N.S.) **65**, 461–463 (1949). (Russian)

By superposition of solutions of torsion problems for thin-walled (tubular) rods of square cross section, twisted by forces distributed along the intersections of the lateral faces (deduced by the author in earlier papers [same Doklady (N.S.) **62**, 305–308 (1948); **65**, 283–286 (1949); these Rev. **10**, 341, 651]) the stress distribution is obtained on the faces of a square rod, twisted by forces applied along a pair of diametrically opposite edges. *I. S. Sokolnikoff.*

Parkus, H. Die Torsion der Kreiswelle mit rechteckiger Längsnut. Anz. Öster. Akad. Wiss. Wien. Math.-Nat. Kl. **85**, 77–79 (1948).

This is a brief study on the torsional stresses existing on the boundary of a rectangular keyway in a circular shaft. The author exhibits three harmonic functions which are said to satisfy the conditions of the problem. The only result stated is to the effect that rounded fillets at the re-entrant corners of the keyway having a radius one eighth of the breadth of the groove reduce the stress concentration in the groove to a constant value which is twice the normal stress value if no keyway were present. *D. L. Holl.*

Vaĭnberg, D. V. The local stresses in a plane ring-shaped disk under two concentrated forces. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 151–158 (1949). (Russian)

The paper is concerned with a detailed analysis of local stresses in a plane ring-shaped elastic disk compressed by two concentrated forces applied to the interior and exterior boundaries of the ring. The solution is based on the determination of two analytic functions of a complex variable and follows the methods of solution of plane problems of elasticity developed by N. I. Muskhelišvili. As a limiting case, the known solution of the corresponding problem for an infinite strip is obtained. *I. S. Sokolnikoff.*

Položiĭ, G. N. A new method for the solution of some mixed problems of the plane theory of elasticity. Doklady Akad. Nauk SSSR (N.S.) **66**, 177–180 (1949). (Russian)

The author deduces a pair of functional equations for the determination of two analytic functions $\varphi(z)$ and $\psi(z)$ of a complex variable $z=x+iy$, giving the solution of the mixed boundary value problem of two-dimensional isotropic elasticity for a closed simply connected region R bounded by a contour C having a finite number of angular points. On the boundary C the component $\nu(s)$ (s being the arc parameter of C) of the displacement vector in the direction of the normal to C and the tangential component $T(s)$ of the applied force are specified. This mixed boundary value problem was first solved by N. I. Muskhelišvili [C. R. (Doklady) Acad. Sci. URSS (N.S.) **4** (1934 III), 141–144] for regions that can be mapped conformally on a unit circle $|z|\leq 1$ by rational functions. If the region R is bounded by a rectilinear polygon, the equations for the determination of φ and ψ simplify to read:

$$\begin{cases}\Im[(k+1)\varphi'(z)]=-2\mu(d\nu/ds)+T(s),\\ \Im[\psi'(z)e^{i\alpha}]=T(s)-\Im[z\varphi''(z)e^{i\alpha}],\end{cases}$$

on C, where k and μ are the elastic constants of the medium, α is the angle formed by the exterior normal to C with the X-axis, and the symbol $\Im$ denotes the imaginary part of the expression following it. The functions $\varphi(z)$ and $\psi(z)$ can be determined with the aid of the formula of Schwarz and by utilizing the continuity properties of stresses and displacements. As an illustration the author solves this mixed boundary value problem for the rectangular region. This solution is new. *I. S. Sokolnikoff* (Los Angeles, Calif.).

Položiĭ, G. N. On a method for the solution of certain mixed problems of the theory of thin plates. Doklady Akad. Nauk SSSR (N.S.) **66**, 353–356 (1949). (Russian)

It is shown in this note that the problem of deflection of a thin isotropic elastic plate bounded by a rectilinear polygon, by normal loads when the displacement and the bending moment are specified along the boundary, can be reduced to a successive determination of two analytic functions of a complex variable from their values on the boundary. To determine one of these functions it is necessary to solve the Dirichlet problem, and the other appears as the solution of the problem of Hilbert. As an example of the method, the author solves the problem of deflection of a simply supported rectangular plate. The well-known elliptic function mapping the rectangle conformally on a unit circle is used in the solution. The ideas involved in this paper are closely related to those appearing in the author's earlier paper [see the preceding review]. *I. S. Sokolnikoff.*

Leggett, D. M. A., and Hopkins, H. G. Sandwich panels and cylinders under compressive end loads. Ministry of Supply [London], Aeronaut. Res. Council, Rep. and Memoranda no. 2262(6134), 22 pp. (1949).

The paper deals with some stability problems concerning plates and shells composed of two thin stiff outer layers and one thick, much less stiff, inner layer. To be taken into

account is the effect of transverse shearing stress deformation in the inner layer on the stability characteristics of the composite structure. This is done approximately in Rayleigh-Ritz fashion. The problems considered are (1) rectangular simply supported flat plate, (2) rectangular simply supported, curved panel, (3) axially symmetric buckling of circular cylindrical shell, (4) flattening of cross-section of originally straight tube with circular cross section under the influence of bending moments applied to the ends of the tube. In each case the resultant formulas are evaluated numerically for significant ranges of parameter values.

E. Reissner (Cambridge, Mass.).

*Reissner, Eric. **On the theory of thin elastic shells.** Reissner Anniversary Volume, Contributions to Applied Mechanics, pp. 231–247. J. W. Edwards, Ann Arbor, Michigan, 1948. $6.50.

General equations for the symmetrical finite deflection of a rotationally symmetric thin shell are first obtained. For small deflections these equations are reduced to a pair of equations for the change of slope of the shell surface and the product of the undeformed radius of the shell to the radial stress. This choice of dependent variable is shown to be advantageous. Two cases of shallow shells give particularly simple solutions: parabolic shells of nth degree and uniform thickness, and parabolic shells of mth degree with thickness varying as mth power of the distance ξ from the apex. For the first case, the solutions can be expressed in terms of cylinder functions; for the second, in powers of ξ. The paper concludes with a discussion of the asymptotic solutions for small thickness. *H. S. Tsien.*

Hildebrand, F. B., Reissner, E., and Thomas, G. B. Notes on the foundations of the theory of small displacements of orthotropic shells. Tech. Notes Nat. Adv. Comm. Aeronaut., no. 1833, 59 pp. (1949).

First, the equations of several known theories of the small displacements of thin shells are derived anew and generalized to the case where the normal to middle surface of the shell is a preferred elastic axis. The following theories are discussed, but the authors do not attempt to present all existing theories. (I) Love's first approximation, which is based upon assumptions permitting (A) the tangential displacements to be approximated as linear functions of the normal distance ζ from the middle surface, (B) the normal displacement to be approximated as independent of ζ, (C) all stresses acting across a surface ζ=constant to be neglected in the stress-strain relations, and (D) ζ/r to be neglected in all expressions, where r is any radius of normal curvature. (II) The theory resulting when assumption (D) is relinquished [R. Byrne, Univ. California Publ. Math. (N.S.) 2 [No. 1, Seminar Rep. in Math. (Los Angeles)], 103–152 (1944); these Rev. 5, 250]; for the case of shells of revolution, this theory was analyzed by the reviewer [Trans. Amer. Math. Soc. 58, 96–166 (1945), see §§ 4, 5, 8; these Rev. 7, 231]. (III) Love's second approximation in which assumption (D) is relinquished and (B) is replaced by the assumption that the normal displacement is a quadratic function of ζ, and the stress σ_{tt} is not neglected. (IV) Basset's theory [Philos. Trans. Roy. Soc. London. Ser. A. 181, 433–480 (1890)] in which all displacements are approximated by quadratic functions of ζ. (V) Trefftz's theory [Z. Angew. Math. Mech. 15, 101–108 (1935)], where all stresses acting upon a section normal to the middle surface are approximated by linear functions of ζ, leading to the same first approximation as Love's theory, but a slightly different second approximation. (VI) Synge and Chien's general theory [von Kármán Anniversary Volume, pp. 103–120, 1941; these Rev. 3, 30] in which all stresses and displacements are expanded in power series in ζ.

The authors carefully derive theories (I), (II) and (III) from the three-dimensional theory and the added assumptions. The discussion of the remaining theories is less detailed. In the latter half of the paper, the authors attempt to formulate a theory of shells in which there are a sufficient number of unknowns in order that physically desirable boundary conditions can be satisfied, avoiding the difficulty analogous to that observed by Kirchhoff in the theory of plates. First, they adopt assumptions (A) and (B), but abandon (C), and introduce two new auxiliary average quantities in addition to the 10 used by Love. The authors expect that their resulting systems of five equations should be such that five resultants can be prescribed upon an edge, but they observe an inconsistency with the theory of flat plates and thus prefer to abandon this structure and start afresh. They then propose to approximate all three displacements by quadratic functions of ζ, and to apply the variational principle of minimum potential energy, so as to simultaneously obtain appropriate boundary conditions. Elaborate calculations now lead to 18 resultants instead of Love's 10. The authors derive a system of equations for the 9 displacement functions, and 9 appropriate boundary conditions, of which they suggest that two may be neglected. They discuss the manner in which the boundary conditions coalesce when the further approximations of Love's theory are introduced. They give no examples illustrating their theory. *C. Truesdell* (Washington, D. C.).

Muštari, H. M. A qualitative investigation of the stressed state of an elastic shell under small deformations and arbitrary displacements. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 121–134 (1949). (Russian)

The paper is largely concerned with the simplification in the formulation of the theory of elastic shells (based on Hooke's law) and with the classification of problems of linear and nonlinear theories of shells. The classification is based on the following conventions. (a) The shell is called thin if its thickness h is of the order of the maximum elongation ϵ permitted by Hooke's law. In this case the quantities of the order h are neglected in comparison with unity. (b) The shell is of medium thickness if $h=O(\epsilon^2)$. In this case terms of the order h^2 are neglected in comparison with unity. (c) The shell is a membrane if $h=O(\epsilon^3)$. Edge effects and some problems involving stressed shells of medium thickness are considered. The author's classification contains considerably fewer types than that proposed by W. Z. Chien [Quart. Appl. Math. 1, 297–327 (1944); 2, 43–59, 120–135 (1944); these Rev. 5, 195, 250; 6, 81].

I. S. Sokolnikoff (Los Angeles, Calif.).

Markov, A. N. The dynamical stability of anisotropic cylindrical shells. Akad. Nauk SSSR. Prikl. Mat. Meh. 13, 145–150 (1949). (Russian)

The author investigates the problem of dynamical stability of thin orthotropic cylindrical shells subjected to pulsating loads. The formulation of the problem is based on Love's approximate theory and assumes that the changes of curvature are adequately characterized by the displacement in the direction of the normal to the surface. The author's formulas for critical loads specialize to known results in the isotropic case. *I. S. Sokolnikoff.*

Alumyaè, N. A. Differential equations of equilibrium of thin elastic shells in the buckling stage. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 95–106 (1949). (Russian)

The object of this paper is to give a simple derivation of approximate differential equations characterizing the equilibrium of thin elastic shells in the buckled state. The following hypotheses are used in the derivation: (1) the deformations remain small even though the displacements in the buckled state are finite; (2) the distribution of deformations along the thickness of the shell follows the Kirchhoff-Love hypothesis; (3) the stressed state of the shell is assumed to be free of moments. The paper contains a system of differential equations for the determination of critical loads and a qualitative discussion of the form of deformation after the loss of stability. The author specializes his equations for the case of thin shallow shells and obtains equations identical with those deduced by H. Tsien, T. von Kármán and L. H. Donnell for flat plates and circular cylindrical shells. Equations deduced by W. Z. Chien in his detailed analysis of thin plates and shells [Quart. Appl. Math. **1**, 297–327 (1944); **2**, 43–59, 120–135 (1944); these Rev. **5**, 195, 250; **6**, 81] also specialize to those given in this paper. *I. S. Sokolnikoff.*

Prokopov, V. K. The equilibrium of an elastic axially-symmetrically loaded thick-walled cylinder. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 135–144 (1949). (Russian)

The object of the paper is to construct solutions suitable for the investigation of the equilibrium of axially-symmetrically loaded thick-walled circular tubes of finite length. In case of axially-symmetric deformations, the displacements and stresses are expressible in terms of one biharmonic function. The author constructs such a function in the form involving Bessel's and modified Bessel's functions of complex arguments, which corresponds to the absence of external forces on the lateral surfaces of the tube. He makes use of this solution to study the equilibrium of a moderately thick cylindrical shell with undeformable ends, subjected to an internal pressure. The results are compared with those given by the usual theory of shells based on the Kirchhoff-Love hypothesis and are in good agreement when the tube is thin.
I. S. Sokolnikoff (Los Angeles, Calif.).

Moufang, R. Das plastische Verhalten von Rohren unter statischem Innendruck bei verschwindender Längsdehnung im Bereich endlicher Verformungen. Ing.-Arch. **12**, 265–283 (1941).

The problem defined in the title of the paper is discussed on the basis of a stress-strain law of the deformation type. The author does not investigate whether the general conditions for the validity of this type of stress-strain law are fulfilled. *W. Prager* (Providence, R. I.).

Ševčenko, K. N. Approximate solution in closed form of a plane elastic-plastic problem with axial symmetry. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 323–328 (1949). (Russian)

Using the stress-strain relations of Hencky, the author gives an approximate analysis of the stresses and strains in a thick walled elastic-plastic cylinder under interior and exterior pressure. [Since these stress-strain relations are of the deformation type, their use is justified only if the general conditions for the validity of stress-strain relations of this type are satisfied. The author does not offer any proof that this is the case.] *W. Prager.*

Sokolovskiĭ, V. V. The equations of plastic equilibrium for a plane stressed state. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 219–221 (1949). (Russian)

In several earlier papers and in his book on plasticity, the author has discussed the equations characterising statically determinate states of plane stress in a perfectly plastic material which obeys the yield condition of von Mises. It was shown that, contrary to what is the case for plane strain, these equations may be of hyperbolic or elliptic type. To a certain extent the hyperbolic case was discussed in the earlier papers, but not the elliptic case. In the present paper a unified theory is developed which covers both cases. While general representation formulas are established no specific examples are treated. *W. Prager* (Providence, R. I.).

Sokolovskiĭ, V. V. Approximate integration of the equations of a plane problem of the theory of plasticity. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 321–322 (1949). (Russian)

The author shows that under certain conditions the basic equations of two-dimensional plasticity can be written approximately so as to permit integration in closed form. This approximation is based on the replacement of $\exp(\xi+\eta)$ by an expression of the form $[A(\xi+\eta+a)]^{2n}$, where ξ, η are the characteristic parameters and A, a, n are constants. Obviously, this is possible only if the range of $\xi+\eta$ happens to be sufficiently small for the problem under consideration.
W. Prager (Providence, R. I.).

Galin, L. A. The elastic-plastic torsion of prismatic bars. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 285–296 (1949). (Russian)

The paper constitutes a systematic study of V. V. Sokolovskiĭ's inverse method of obtaining distributions of shearing stress which arise in the elastic-plastic torsion of prismatic bars [Appl. Math. Mech. [Akad. Nauk SSSR. Prikl. Mat. Mech.] **6**, 241–246 (1942); these Rev. **4**, 180]. It is assumed that the plastic region completely encloses the elastic region. In view of the yield condition, the components of shearing stress in the plastic region are then written in the form $\tau_x=k\sin\psi(t)$, $\tau_y=k\cos\psi(t)$, where k denotes the yield stress in simple shear and ψ is a function of the parameter t. The contour of the cross section and the elastic-plastic stress distribution are represented in terms of the function $\psi(t)$. The solution of the direct problem is reduced to the solution of a nonlinear integral equation for $\psi(t)$. The following special cases of the inverse problem are discussed: $\psi(t)=t$ (circle) and piecewise constant $\psi(t)$ (nearly polygonal cross section). *W. Prager* (Providence, R. I.).

Hemp, W. S. On the natural frequencies of a reinforced circular cylinder. Coll. Aeronaut. Cranfield. Rep. no. **26**, i+8 pp. (1949).

This note attempts to evaluate the effects of shear deflection on the stiffness, and hence on the lower natural frequencies of a cantilever beam in the form of a reinforced circular cylindrical shell. The shell is reinforced by rigid rings so that plane cross-sections remain plane in bending. The results are in agreement with the usual strength-of-materials beam considerations in this connection.
M. Goland (Kansas City, Mo.).

Sengupta, A. M. Note on a simple case of forced torsional oscillation of a circular cylinder. Bull. Calcutta Math. Soc. **41**, 47–48 (1949).

One end of circular elastic cylinder is fixed, the cylindrical surface is free from stress and to the other end is applied a

shear traction proportional to the distance from the axis and trigonometric in time. The displacement is given in terms of a double trigonometric series. *G. F. Carrier.*

Lenskiĭ, V. S. On the elastic-plastic impact of a rod on a rigid obstacle. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 165–170 (1949). (Russian)

A cylindrical rod of elastic-plastic material with a linear work hardening law is considered to strike a rigid target normally. The resulting system of elastic and plastic longitudinal waves of plane stress is analysed. The elastic waves are repeatedly reflected at the ends of the rod, leading to multiple interactions with the plastic wave front emanating from the point of impact, which produce bands of constant plastic strain adjacent to the impact end. The strain distribution and time of impact are obtained as a function of the impact velocity. The limitation stated by the author of particle velocities small compared with wave velocities can be avoided by the use of Lagrange coordinates. The use of wave-front trajectories in the (x, t)-plane would simplify the book-keeping of the multiple interactions [cf. White, J. Appl. Mech. **16**, 39–52 (1949)]. *E. H. Lee.*

Lunc, Ya. L. On the propagation of spherical waves in an elastic-plastic medium. Akad. Nauk SSSR. Prikl. Mat. Meh. **13**, 55–78 (1949). (Russian)

The author considers the propagation of a pulse of pressure occurring at the surface of a spherical cavity in an elastic-plastic medium. A work hardening law involving a general relation between the second order invariants of the stress and strain tensors is assumed, which in this case includes a flow type of law because of the symmetry of the system. The equation of motion, which represents both elastic and plastic spherical waves, is developed in terms of independent variables based on the characteristics. The propagation of the wave due to a given pulse of increasing pressure is determined for a linear work hardening law. The propagation of an elastic unloading wave due to a subsequent decrease of the pressure in the cavity is also analysed. *E. H. Lee* (Providence, R. I.).

Pastori, M. Propagazione ondosa nei continui anisotropi e corrispondenti direzioni principali. Nuovo Cimento (9) **6**, 187–193 (1949).

This paper concerns the propagation of waves (surfaces across which the velocity is continuous but the velocity gradient suffers a discontinuity) in anisotropic elastic media, where the stress p_{rs} and the infinitesimal strain ξ^{ik} are related by the generalized Hooke's law $p_{rs}=c_{rsik}\xi^{ik}$. Let n^j be the wave normal, let ρ be the density, and let the propagation tensor A_{ik} be given by $-\rho A_{ik}\equiv c_{ijhk}n^jn^h$. The classical result of Hadamard [Leçons sur la propagation des ondes, Hermann, Paris, 1903, pp. 245, 252] states that for a wave with given normal n^j, the discontinuity vector must be parallel to one of the three (or possibly infinitely many) principal directions of A_{ik}, and that to each of these there corresponds a definite real speed of propagation. The author proposes that instead of considering the wave normal as given, one may simply investigate the intrinsic properties of the elastic medium. To this end he introduces the symmetric tensor B_{rs} defined by $-\rho B_{rs}\equiv c^i_{rts}$, whose principal directions he calls the principal directions of the medium at the point in question, and proves that for these directions the sum of the squares of the three speeds of propagation has a stationary value relative to other directions. When the medium is of Green's type, in which the elastic tensor reduces to the form $c_{ijhk}=-Ca_{ij}a_{hk}+c^{jh}(\epsilon_{jis}\epsilon_{hrk}+\epsilon_{jrs}\epsilon_{hri})$, where C is a scalar, a_{ij} is the metric tensor, and c^{jh} is a symmetrical tensor, then the principal directions of the medium coincide with the principal directions of c^{jh}, which are the principal directions of Fresnel's ellipsoid. The paper closes with the remark that all invariants of A_{ij} are symmetric functions of the three speeds of propagation for the direction in question, while all invariants of B_{rs} are symmetric functions of the three sums of squares of the three speeds of propagation for the three principal directions. *C. Truesdell* (Washington, D. C.).

Päsler, Max. Die Differentialgleichung für den zeitlichen Verlauf von thermisch gedämpften elastischen Schwingungen. Ann. Physik (6) **4**, 14–24 (1948).

Thermo-elastic equations of motion are derived with a heat transfer term resulting from the small temperature changes associated with dilatation and contraction. A small but possibly significant damping is thus introduced. Writing the displacement vector as a product of a function of time F and a function of position leads to a linear differential equation of third order with constant coefficients for F. A discussion is given of the various moduli which enter the problem. *D. C. Drucker* (Providence, R. I.)

Federhofer, Karl. Eigenschwingungen von geraden Stäben mit dünnwandigen und offenen Querschnitten. Österreich. Akad. Wiss. Math.-Nat. Kl. S.-B. IIa. **156**, 393–416 (1948).

A study is made of the natural vibrations of a uniform beam with a thin-walled open cross-section. The author first points out that a beam can exhibit natural vibrations in bending alone only when the shear-center and center-of-gravity of its cross-section are coincident; then the beam vibration spectrum displays uncoupled bending vibrations in its two principal bending directions, and also pure torsional vibrations (which are uncoupled from the bending mechanism). However, when the shear-center and center-of-gravity are not coincident, then the natural modes of the beam must contain both coupled bending and torsional motions. This coupling between bending and torsion becomes important in the present case.

The general problem where the shear-center and center-of-gravity are not coincident is then investigated analytically. In the development, the elementary theory of elastic bending is utilized, while the beam torsion is accounted for by the theory of R. Kappus [Luftfahrtforschung **14**, 444–457 (1937)] which applies to cross-sections with open profiles. The three differential equations which govern the transverse displacement and the twist of the undistorted beam sections are derived by the energy method; this, of course, also yields the boundary conditions which are appropriate for the beam ends. The character of the solution for these equations is discussed in some detail, both for beams with simply-supported and clamped ends. The solution is next specialized to the case of beams with cross-sections having a single axis of symmetry. Such a beam has one series of modes involving pure bending, and another entailing coupled bending and torsion. The lower coupled modes are investigated in some detail, and results are given for a series of beams with channel cross-sections. The effects of the bending and torsion coupling are found to reduce as the beam is made longer, and as the cross-section is made thicker. *M. Goland* (Kansas City, Mo.).

Lapwood, E. R. The disturbance due to a line source in a semi-infinite elastic medium. Philos. Trans. Roy. Soc. London. Ser. A. **242**, 63–100 (1949).

The intricate diffraction effects accompanying the reflection of a cylindrical wave-pulse at the plane free surface of an isotropic elastic solid are re-examined in this paper. The source is distributed uniformly along an infinite line within the solid, parallel to the surface, and emits a pulse in the form of a Heaviside unit function. Both dilatational (P) and (S) pulses are considered. The solution is given in terms of double integrals and the bulk of the paper is devoted to their interpretation, with special attention to the contributions of the poles. Exact evaluations are not possible, but approximations reveal the main features including some about which there has been some uncertainty. When the distances of the source (F) and the point of arrival (G) from the surface are small in comparison with their distance apart it is found that: (a) for an initial P-pulse at F, the pulses at G are, in order of arrival, direct P, reflected P, reflected S, surface S, secondary reflected S and Rayleigh surface; (b) for an initial S-pulse at F, the sequence of pulses arriving at G is reflected P, surface P, direct S, secondary reflected P, reflected S and Rayleigh surface. The surface and secondary pulses represent the diffraction phenomena due to curvature of the incident wave fronts, while the remaining pulses are the same as those which would arise from incident plane waves. *R. D. Mindlin.*

Scholte, J. G. On the large displacements commonly regarded as caused by Lovewaves and similar dispersive surface-waves. I, II. Nederl. Akad. Wetensch., Proc. **51**, 533–543, 642–649 (1948).

These two installments correspond to the author's chapters I to III of a five chapter treatise. The author summarizes his work as follows.

In this paper an attempt has been made to calculate the motion caused by a disturbance taking place in a stratified medium. The method used in a previous paper on this subject started with a Fourier-analysis of the time factor of the initial disturbance; the motion connected with each component of the Fourier-series was calculated and the final result was reached by adding up these partial movements. This method, however, yields results which are not entirely interpretable. The first difficulty one meets when trying to correlate these results to the observed properties of seismograms lies in the fact that the time of arrival of the surface-waves cannot be calculated; as these waves form the major part of the movement at large distances this impossibility is very disconcerting. A second difficulty is scarcely less important: as these surface-waves satisfy the frequency equation of the system (for instance the Love-equation) their phase-velocity is a function of the frequency; it follows that the surface-waves have to show the usual dispersion properties. But the observed relation between the velocity and the period of the large displacements is in direct contradiction to the dispersion theory.

These considerations have led to an investigation of the surface-waves, which is described in chapter I; it appears that it is by no means a priori certain that the surface-waves which satisfy the frequency equation occur in seismic disturbances. Consequently it is possible that a method which obtains these waves as the predominant terms of a series of movements is not adequate to the problem. It is therefore necessary to use another method in order to ascertain whether these surface-waves are relevant or only the results of an inadequate method. By means of the method of Carson it is possible to avoid the Fourier-analysis of the initial disturbance; this method, applied in chapters II and V, leads to results which are easily interpreted as they contain no dispersive surface-waves.

It is difficult to follow step by step the physical implications of the author's treatment. There are a number of typographical errors; but they are not troublesome. [Cf. the two following reviews.] *J. B. Macelwane.*

Scholte, J. G. On the large displacements commonly regarded as caused by Love-waves and similar dispersive surface-waves. III. Nederl. Akad. Wetensch., Proc. **51**, 828–835 (1948).

[Cf. the preceding review.] Conditions for a purely longitudinal movement are assumed. The author refers to the displacements taking place under the postulated conditions as motion in an ocean caused by a subaqueous explosion although he does not include the Ewing bubble effect, density-temperature channeling, and other phenomena characteristic of a physical ocean. The structure postulated by the author is a homogeneous liquid layer of finite thickness resting on a solid body of semi-infinite extent. The primary disturbance is first assumed to occur in a plane within the liquid layer parallel to the interface. The results are then transformed to correspond to a point disturbance within the liquid layer by a Laplace transformation applied to the displacement potential. A Stoneley function is found which determines a major part of the disturbance.

J. B. Macelwane (St. Louis, Mo.).

Scholte, J. G. On the large displacements commonly regarded as caused by Love-waves and similar dispersive surface-waves. IV. Nederl. Akad. Wetensch., Proc. **51**, 969–976 (1948).

[Cf. the two preceding reviews.] In the first paragraph the author supposes a disturbance in the liquid layer propagated downward into the second medium and considers that cone of ray paths which will impinge on the inter-surface at the critical angle or greater so as to set up boundary and reflected waves. The corresponding Laplace functions are analyzed for a series of conditions and time intervals. The second paragraph begins with a statement concerning a supposed seismological definition of a surface which would be conceded by few, if any, seismologists. The paragraph is largely devoted to a discussion of Cagniard's theoretical work and of supposed results from observation.

J. B. Macelwane (St. Louis, Mo.).

Scholte, J. G. On true and pseudo Rayleigh waves. Nederl. Akad. Wetensch., Proc. **52**, 652–653 (1949).

The author discusses solutions by himself, in previously published papers, and by Cagniard [Reflexion et refraction des ondes seismiques progressives, Gauthier-Villars, Paris, 1939] for the formation and propagation of boundary waves of the Rayleigh and Stoneley type. He states that his previous criticism of the root of the equation of condition $D = 0$ obtained by Cagniard was not justified and that there are really two roots, one of which is very important when one of the two semi-infinite media is a fluid of low density like the atmosphere, and the other is solid, while the second root is the important one when both are solid or one is solid and the other is liquid like the ocean. The first was obtained by Cagniard and the second by the author. *J. B. Macelwane* (St. Louis, Mo.).